

Weighted norm inequalities, spectral multipliers and Littlewood-Paley operators in the Schrödinger settings

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Abstract In this paper, we establish a good- λ inequality with two parameters in the Schrödinger settings. As it's applications, we obtain weighted estimates for spectral multipliers and Littlewood-Paley operators and their commutators in the Schrödinger settings.

1. Introduction

In this paper, we consider the divergence Schrödinger differential operator

$$L = -\partial_i(a_{ij}(x)\partial_j) + V(x) \text{ on } \mathbb{R}^n, \quad n \geq 3,$$

where $V(x)$ is a nonnegative potential satisfying certain reverse Hölder class. In this paper, we always assume that the coefficients of these operators are bounded and measurable, and a_{ij} are real symmetric, uniformly elliptic, i.e., for some $\delta \in (0, 1]$,

$$a_{ij} = a_{ji}, \quad |a_{ij}| \leq \delta^{-1}, \quad \delta|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \delta^{-1}|\xi|^2. \quad (1.1)$$

We say a nonnegative locally L^q integral function $V(x)$ on \mathbb{R}^n is said to belong to $B_q(1 < q \leq \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x, r)|} \int_{B(x, r)} V^q(y) dy \right)^{1/q} \leq C \left(\frac{1}{|B(x, r)|} \int_{B(x, r)} V(y) dy \right)$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $B(x, r)$ denotes the ball centered at x with radius r . In particular, if V is a nonnegative polynomial, then $V \in B_\infty$. Throughout this paper, we always assume that $0 \not\equiv V \in B_n/2$.

The study of Schrödinger operator $L_0 = -\Delta + V$ recently attracted much attention; see [2, 5, 6, 14, 21, 25, 26]. In particular, it should be pointed out that Shen [21] proved the Schrödinger type operators, such as $\nabla(-\Delta + V)^{-1}\nabla$, $\nabla(-\Delta + V)^{-1/2}$, $(-\Delta + V)^{-1/2}\nabla$

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with $V \in B_n$, $(-\Delta + V)^{i\gamma}$ with $\gamma \in \mathbb{R}$ and $V \in B_{n/2}$, are standard Calderón-Zygmund operators. Later, Auscher and Ali [2] improved some results of Shen [21].

Recently, Bongioanni, etc, [5] proved $L^p(\mathbb{R}^n)$ ($1 < p < \infty$) boundedness for commutators of Riesz transforms associated with Schrödinger operator with $BMO_\theta(\rho)$ functions which include the class BMO function, and they [6] established the weighted boundedness for Riesz transforms, fractional integrals and Littlewood-Paley functions associated with Schrödinger operators with weight $A_p^{\rho, \theta}$ class which includes the Muckenhoupt weight class. Very recently, Tang [23, 24] established a new good- λ inequality, and obtained the weighted norm inequalities for some Schrödinger type operators, which include commutators for Riesz transforms, fractional integrals and Littlewood-Paley functions associated with Schrödinger operators.

It should be pointed out that the results above were obtained by using sizes estimates of kernels of Schrödinger type operators. In some cases, we may meet some Schrödinger operators that they do not have an integral representation by a kernel with sizes estimates. To deal with latter case, in this paper, we will establish a good- λ inequality with two parameters in the Schrödinger settings. As it's applications, we obtain weighted estimates for spectral multipliers and Littlewood-Paley operators and their commutators in the Schrödinger settings.

The paper is organized as follows. In Section 2, we give some notation and basic results. In Section 3, we establish a good- λ inequality with two parameters in the Schrödinger settings. In Section 4, we obtain weighted inequalities for spectral multipliers and their commutators in the Schrödinger settings. Finally, we give weighted estimates for Littlewood-Paley operators and their commutators in the Schrödinger settings. in Section 5.

Throughout this paper, we let C denote constants that are independent of the main parameters involved but whose value may differ from line to line. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $1/C \leq A/B \leq C$.

2. Preliminaries

We first recall some notation. Given $B = B(x, r)$ and $\lambda > 0$, we will write λB for the λ -dilate ball, which is the ball with the same center x and with radius λr . Similarly, $Q(x, r)$ denotes the cube centered at x with the sidelength r (here and below only cubes with sides parallel to the coordinate axes are considered), and $\lambda Q(x, r) = Q(x, \lambda r)$. Given a Lebesgue measurable set E and a weight ω , $|E|$ will denote the Lebesgue measure of E and $\omega(E) = \int_E \omega dx$. For $0 < p < \infty$

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(y)|^p \omega(y) dy \right)^{1/p}, \quad \|f\|_{L^{p,\infty}(\omega)} = \sup_{\lambda > 0} \lambda^{1/p} \omega\{|f(y)| > \lambda\}.$$

If $\omega = 1$, we simply denote $\|f\|_{L^p} = \|f\|_{L^p(\omega)}$, $\|f\|_{L^{p,\infty}} = \|f\|_{L^{p,\infty}(\omega)}$.

The function $m_V(x)$ is defined by

$$\rho(x) = \frac{1}{m_V(x)} = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y) dy \leq 1 \right\}.$$

Obviously, $0 < m_V(x) < \infty$ if $V \neq 0$. In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \sim (1 + |x|)$ with $V = |x|^2$.

Lemma 2.1([21]). *There exists $l_0 > 0$ and $C_0 > 1$ such that*

$$\frac{1}{C_0} (1 + |x - y|/\rho(x))^{-l_0} \leq \frac{\rho(y)}{\rho(x)} \leq C_0 (1 + |x - y|/\rho(x))^{l_0/(l_0+1)}.$$

In particular, $\rho(x) \sim \rho(y)$ if $|x - y| < C\rho(x)$.

In this paper, we write $\Psi_\theta(B) = (1 + r/\rho(x_0))^\theta$, where $\theta > 0$, x_0 and r denotes the center and radius of B respectively.

A weight will always mean a nonnegative function which is locally integrable. As in [6], we say that a weight ω belongs to the class $A_p^{\rho,\theta}$ for $1 < p < \infty$, if there is a constant C such that for all ball $B = B(x, r)$

$$\left(\frac{1}{\Psi_\theta(B)|B|} \int_B \omega(y) dy \right) \left(\frac{1}{\Psi_\theta(B)|B|} \int_B \omega^{-\frac{1}{p-1}}(y) dy \right)^{p-1} \leq C.$$

We also say that a nonnegative function ω satisfies the $A_1^{\rho,\theta}$ condition if there exists a constant C for all balls B

$$M_{\rho,\theta}(\omega)(x) \leq C\omega(x), \text{ a.e. } x \in \mathbb{R}^n.$$

where

$$M_{\rho,\theta}f(x) = \sup_{x \in B} \frac{1}{\Psi_\theta(B)|B|} \int_B |f(y)| dy.$$

Since $\Psi_\theta(B) \geq 1$, obviously, $A_p \subset A_p^{\rho,\theta}$ for $1 \leq p < \infty$, where A_p denote the classical Muckenhoupt weights; see [15] and [17]. We will see that $A_p \subset A_p^{\rho,\theta}$ for $1 \leq p < \infty$ in some cases. In fact, let $\theta > 0$ and $0 \leq \gamma \leq \theta$, it is easy to check that $\omega(x) = (1 + |x|)^{-(n+\gamma)} \notin A_\infty = \bigcup_{p \geq 1} A_p$ and $\omega(x)dx$ is not a doubling measure, but $\omega(x) = (1 + |x|)^{-(n+\gamma)} \in A_1^{\rho,\theta}$ provided that $V = 1$ and $\Psi_\theta(B(x_0, r)) = (1 + r)^\theta$.

When $V = 0$ and $\theta = 0$, we denote $M_{0,0}f(x)$ by $Mf(x)$ (the standard Hardy-Littlewood maximal function). It is easy to see that $|f(x)| \leq M_{\rho,\theta}f(x) \leq Mf(x)$ for a.e. $x \in \mathbb{R}^n$ and $\theta \geq 0$.

Similar to the classical Muckenhoupt weights(see [15, 22]), we give some properties for weight class $A_p^{\rho,\theta}$ for $p \geq 1$.

Proposition 2.1. *Let $\omega \in A_p^{\rho,\infty} = \bigcup_{\theta \geq 0} A_p^{\rho,\theta}$ for $p \geq 1$. Then*

(i) *If $1 \leq p_1 < p_2 < \infty$, then $A_{p_1}^{\rho,\theta} \subset A_{p_2}^{\rho,\theta}$.*

(ii) $\omega \in A_p^{\rho, \theta}$ if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}^{\rho, \theta}$, where $1/p + 1/p' = 1$.

(iii) If $\omega \in A_p^{\rho, \infty}$, $1 < p < \infty$, then there exists $\epsilon > 0$ such that $\omega \in A_{p-\epsilon}^{\rho, \infty}$.

Proof. (i) and (ii) are obvious by the definition of $A_p^{\rho, \theta}$. (iii) is proved in [6]. In fact, from Lemma 5 in [6], we know that if $\omega \in A_p^{\rho, \theta}$, then $\omega \in A_{p_0}^{\rho, \theta_0}$, where $p_0 = 1 + \frac{p-1}{\delta} < p$ with some $\delta > 1$ (δ is a constant depending only on the RH_δ^{loc} constant of ω , see below) and $\theta_0 = \frac{\theta p + \eta(p-1)}{p_0}$ with $\eta = \theta p + (\theta + n)\frac{p l_0}{l_0 + 1} + (l_0 + 1)n$. \square

The local reverse Hölder classes are defined in the following way: $\omega \in RH_q^{loc}$, $1 < q < \infty$, if there is a constant C such that for every ball $B(x_0, r) \subset \mathbb{R}^n$ with $r < \rho(x_0)$,

$$\left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{1/q} \leq C \frac{1}{|B|} \int_B w(x) dx.$$

The endpoint $q = \infty$ is given by the condition: $\omega \in RH_\infty^{loc}$ whenever, for every ball $B(x_0, r) \subset \mathbb{R}^n$ with $r < \rho(x_0)$,

$$\omega(x) \leq C \frac{1}{|B|} \int_B w(x) dx, \quad \text{for a.e. } x \in B.$$

From Lemma 5 in [6], we know that if $\omega \in A_p^{\rho, \infty}$ for $p \geq 1$, then there exists a $q > 1$, such that $\omega \in RH_q^{loc}$. In addition, it is easy to see that if $\omega \in RH_q^{loc}$ with $1 < q < \infty$, then there exists $\epsilon > 0$ such that $\omega \in RH_{q+\epsilon}^{loc}$.

Next we give some weighted estimates for $M_{\rho, \theta}$.

Lemma 2.2([23]). *Let $1 \leq p_1 < \infty$ and suppose that $\omega \in A_{p_1}^{\rho, \theta}$. If $p_1 < p < \infty$, then the equality*

$$\int_{\mathbb{R}^n} |M_{\rho, \theta} f(x)|^p \omega(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx.$$

Furthermore, let $1 \leq p < \infty$, $\omega \in A_p^\theta$ if and only if

$$\omega(\{x \in \mathbb{R}^n : M_{\rho, \theta} f(x) > \lambda\}) \leq \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx.$$

Lemma 2.3([23]). *Let $1 < p < \infty$, $p' = p/(p-1)$ and assume that $\omega \in A_p^{\rho, \theta}$. There exists a constant $C > 0$ such that*

$$\|M_{\rho, p' \theta} f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}.$$

In addition, Bongioanni, etc, [5] introduce a new space $BMO_\theta(\rho)$ defined by

$$\|f\|_{BMO_\theta(\rho)} = \sup_{B \subset \mathbb{R}^n} \frac{1}{\Psi_\theta(B)|B|} \int_B |f(x) - f_B| dx < \infty,$$

where $f_B = \frac{1}{|B|} \int_B f(y) dy$.

In particularly, Bongioanni, etc, [5] proved the following result for $BMO_\theta(\rho)$.

Lemma 2.4. *Let $\theta > 0$ and $1 \leq s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\left(\frac{1}{|B|} \int_B |b - b_B|^s \right)^{1/s} \leq C_{\theta,s} \|b\|_{BMO_\theta(\rho)} \left(1 + \frac{r}{\rho(x)} \right)^{\theta'},$$

for all $B = B(x, r)$, with $x \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (l_0 + 1)\theta$.

Now we define $BMO_\infty(\rho) = \bigcup_{\theta>0} BMO_\theta(\rho)$. Obviously, the classical BMO is properly contained in $BMO_\theta(\rho)$; more examples see [5].

Applying Lemma 2.4, Tang [23] proved the following John-Nirenberg inequality for $BMO_\theta(\rho)$.

Proposition 2.2. *Suppose that f is in $BMO_\theta(\rho)$. There exist positive constants γ and C such that*

$$\sup_B \frac{1}{|B|} \int_B \exp \left\{ \frac{\gamma}{\|f\|_{BMO_\theta(\rho)} \Psi_{\theta'}(B)} |f(x) - f_B| \right\} dx \leq C.$$

We remark that balls can be replaced by cubes in definitions of $A_p^{\rho,\theta}$, $BMO_\theta(\rho)$ and $M_{\rho,\theta}$, since $\Psi_\theta(B) \leq \Psi_\theta(2B) \leq 2^\theta \Psi_\theta(B)$.

Finally, we recall some basic definitions and facts about Orlicz spaces, referring to [20] for a complete account.

A function $B(t) : [0, \infty) \rightarrow [0, \infty)$ is called a Young function if it is continuous, convex, increasing and satisfies $\Phi(0) = 0$ and $B \rightarrow \infty$ as $t \rightarrow \infty$. If B is a Young function, we define the B -average of a function f over a cube(ball) Q by means of the following Luxemburg norm:

$$\begin{aligned} \|f\|_{B,Q,\omega} &= \inf \left\{ \lambda > 0 : \frac{1}{\omega(Q)} \int_Q B \left(\frac{|f(y)|}{\lambda} \right) \omega(y) dy \leq 1 \right\} \\ &\sim \inf \left\{ t > 0 : t + \frac{t}{\omega(Q)} \int_Q B \left(\frac{|f(y)|}{t} \right) \omega(y) dy \right\}, \end{aligned}$$

where $\omega(y)dy$ is Borel measure. If A , B and C are Young functions such that

$$A^{-1}(t)B^{-1}(t) \leq C^{-1}(t),$$

where A^{-1} is the complementary Young function associated to A , then

$$\|fg\|_{C,Q,\omega} \leq 2\|f\|_{A,Q,\omega}\|g\|_{B,Q,\omega}.$$

The examples to be considered in our study will be $A^{-1}(t) = \log(1+t)$, $B^{-1}(t) = t/\log(e+t)$ and $C^{-1}(t) = t$. Then $A(t) \sim e^t$ and $B(t) \sim t \log(e+t)$, which gives the generalized Hölder's inequality for any cubes(balls) Q

$$\frac{1}{\omega(Q)} \int_Q |fg| \omega(y) dy \leq \|f\|_{A,Q,\omega} \|g\|_{B,Q,\omega}$$

holds for any Borel measure $\omega(y)dy$. And we define the corresponding maximal function

$$M_B f(x) = \sup_{Q: x \in Q} \|f\|_{B,Q} := \sup_{Q: x \in Q} \|f\|_{B,Q,1}$$

and for $0 < \eta < \infty$

$$M_{B,\rho,\eta} f(x) = \sup_{Q: x \in Q} (\Psi_\eta(Q))^{-1} \|f\|_{B,Q}.$$

The examples such as $B(t) = t(1 + \log^+ t)^\alpha$ ($\alpha > 0$) with the maximal function denoted by $M_{L(\log L)^\alpha, \rho, \eta}$. The complementary Young function is given by $\bar{B}(t) \approx e^{t^\alpha}$ with the corresponding maximal function denoted by $M_{\exp L^\alpha, \rho, \eta}$. In this previous case, it is well known that for $k \geq 1$, from the the proof of Lemma 4.1 in [23], we have

$$M_{L(\log L)^k, \rho, \eta} f \leq C_{\eta,k} M_{\rho, \eta/2^k}^{k+1} f, \quad (2.1)$$

where $M_{\rho, \eta/2^k}^{k+1}$ is the $k+1$ -iteration of $M_{\rho, \eta/2^k}$.

For these example and using Theorem 2.1, if $b \in BMO_\eta(\rho)$ and b_Q denotes its average on the cube(ball) Q , then $\|(b - b_Q)/\Psi_{\eta'}(Q)\|_{\exp L, Q} \leq C \|b\|_{BMO_\eta(\rho)}$, where $\eta' = (1 + l_0)\eta > 0$. This yields the following estimates: First, for each cube(ball) Q and $x \in Q$

$$\begin{aligned} & \frac{1}{\Psi(Q)^{\eta'(kp_0+1)}|Q|} \int_Q |b - b_Q|^{kp_0} |f|^{p_0} dy \\ & \leq \|(b - b_Q)/\Psi_{\eta'}(Q)\|_{\exp L, Q}^{kp_0} \| |f|^{p_0} / \Psi_{\eta'}(Q) \|_{L(\log L)^{kp_0}, Q} \\ & \leq C \|b\|_{BMO_\eta(\rho)}^{kp_0} M_{L(\log L)^{kp_0}, \rho, \eta'}(|f|^{p_0})(x) \\ & \leq C \|b\|_{BMO_\eta(\rho)}^{kp_0} M_{\rho, \eta'/2^{[kp_0]+1}}^{[kp_0]+2}(|f|^{p_0})(x). \end{aligned} \quad (2.2)$$

where $[s]$ is the integer part of s . Second, for $j \geq 1$ and each Q ,

$$\begin{aligned} \|(b - b_Q)/\Psi_{\eta'}(2^j Q)\|_{\exp L, \rho, 2^j Q} & \leq \|(b - b_{2^j Q})/\Psi_{\eta'}(2^j Q)\|_{\exp L, \rho, 2^j Q} + |b_{2^j Q} - b_Q|/\Psi_{\eta'}(2^j Q) \\ & \leq Cj \|b\|_{BMO_\eta(\rho)}. \end{aligned} \quad (2.3)$$

In addition, for any cube(ball) $Q = Q(x_0, r)$ with $r < \rho(x_0)$, let $b \in BMO_\theta(\rho)$ and b_Q denotes its average on Q , if $\omega \in RH_q^{loc}$ for some $q > 1$, then by Proposition 2.2,

$$\|(b - b_Q)\|_{\exp L, Q, \omega} \leq C \|b\|_{BMO_\theta(\rho)}, \quad (2.4)$$

and

$$\frac{1}{\omega(Q)} \int_Q |b(y) - b_Q| \omega(y) dy \leq C \|b\|_{BMO_\theta(\rho)}. \quad (2.5)$$

3. Two-parameter good- λ estimates

In this section, we always assume that the auxiliary $\rho(x)$ satisfies Lemma 2.1.

Theorem 3.1. Fix $1 < q \leq \infty$, $a \geq 1$, $\theta > 0$ and $\omega \in RH_{s'}^{loc}$ with $1 \leq s < \infty$ and $1/s + 1/s' = 1$. Then, there exists $C = C(q, n, a, \omega, s, \theta)$ and $K_0 = K_0(n, a) \geq 1$ with the following properties: Assume that F, G, H_1 are non-negative measurable functions on \mathbb{R}^n such that for any cube $Q = Q(x_0, r)$ with $r < \rho(x_0)$, there exist non-negative functions G_{8Q} and H_{8Q} with $F(x) \leq G_{8Q}(x) + H_{8Q}(x)$ for a.e. $x \in 8Q$ and

$$\left(\frac{1}{|8Q|} \int_{8Q} H_{8Q}(y)^q dy \right)^{1/q} \leq a(M_{\rho, \theta} F(x) + H_1(\bar{x})), \quad \forall x, \bar{x} \in 8Q; \quad (3.1)$$

and for any $x \in \mathbb{R}^n$

$$\sup_{x \in Q, r < \rho(x_0)} \frac{1}{|8Q|} \int_{8Q(x_0, r)} G_{8Q}(y) dy + \sup_{x \in Q, r \geq \rho(x_0)} \frac{1}{\Psi_\theta(Q)|Q|} \int_{Q(x_0, r)} |F(y)| dy \leq G(x). \quad (3.2)$$

Then for all $\lambda > 0$, $K \geq K_0$ and $0 < \gamma < 1$

$$\omega\{M_{\rho, \theta} F > K\lambda, G + H \leq \gamma\lambda\} \leq C \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{1/s} \omega\{M_{\rho, \theta} F > \lambda\}. \quad (3.3)$$

As a consequence, for all $0 < p < q/s$, we have

$$\|M_{\rho, \theta} F\|_{L^p(\omega)} \leq C(\|G\|_{L^p(\omega)} + \|H_1\|_{L^p(\omega)}), \quad (3.4)$$

provided $\|M_{\rho, \theta} F\|_{L^p(\omega)} < \infty$, and

$$\|M_{\rho, \theta} F\|_{L^{p, \infty}(\omega)} \leq C(\|G\|_{L^{p, \infty}(\omega)} + \|H_1\|_{L^{p, \infty}(\omega)}), \quad (3.5)$$

provided $\|M_{\rho, \theta} F\|_{L^{p, \infty}(\omega)} < \infty$. Furthermore, if $p \geq 1$ then (3.4) and (3.5) hold, provided $F \in L^1$ (whether or not $M_{\rho, \theta} F \in L^p(\omega)$).

Proof. We borrow some ideas from [3, 23]. It suffices to consider the case $H = G$: indeed, set $\tilde{G} = G + H$. Then (3.1) holds with \tilde{G} in place of H and also (3.2) holds with \tilde{G} in place of G .

Since from on we assume that $H = G$. Set $E_\lambda = \{x \in \mathbb{R}^n : M_{\rho, \theta} F(x) > \lambda\}$ which is assumed to have finite measure (otherwise there is nothing to prove). Clearly, E_λ is an open set. By Whitney's decomposition, there exists a family of pairwise disjoint cubes $\{Q_j\}_j$ so that $E_\lambda = \bigcup_j Q_j$ with the property that $4Q_j$ meets E_λ^c , that is, there exists $x_j \in 4Q_j$ such that $M_{\rho, \theta} F(x_j) \leq \lambda$.

Set $B_\lambda = \{M_{\rho, \theta} F > K\lambda, 2G \leq \gamma\lambda\}$. Since $K \geq 1$ we have that $B_\lambda \subset E_\lambda$. Therefore $B_\lambda \subset \bigcup_j B_\lambda \cap Q_j$. We first claim that if $Q_j = Q(x_j^0, r_j)$ with $r_j \geq \rho(x_j^0)$, then

$$B_\lambda \cap Q_j = \emptyset.$$

In fact, if $r \geq \rho(x_j)$ and $x \in B_\lambda \cap Q_j$, then by (3.2), we have

$$K_0\lambda < \frac{1}{\Psi_\theta(Q_j)|Q_j|} \int_{Q_j} |F(y)| dy \leq G(x) < \gamma\lambda$$

but $\gamma < 1$, hence, the set in question is empty.

Hence, we next only consider these cubes $Q_j = Q(x_j^0, r_j)$ with $r_j < \rho(x_j^0)$. For each j we assume that $B_\lambda \cap Q_j \neq \emptyset$ (otherwise we discard this cube) and so there is $\bar{x}_j \in Q_j$ so that $G(\bar{x}_j) \leq \gamma\lambda/2$. Since $M_{\rho,\theta}F(x_j) \leq \lambda$, there is \tilde{C}_0 depending only on n, θ, l_0, C_0 such that for every $K \geq \tilde{C}_0$ we have

$$\begin{aligned} |B_\lambda \cap Q_j| &\leq |\{M_{\rho,\theta}F > K\lambda\} \cap Q_j| \leq |\{M_{\rho,\theta}(F\chi_{8Q_j}) > (K/\tilde{C}_0)\lambda\}| \\ &\leq |\{M_{\rho,\theta}(G_{8Q_j}\chi_{8Q_j}) > (2K/\tilde{C}_0)\lambda\}| + |\{M_{\rho,\theta}(H_{8Q_j}\chi_{8Q_j}) > (2K/\tilde{C}_0)\lambda\}|, \end{aligned}$$

where we have used $F\chi_{8Q_j} \leq G_{8Q_j}\chi_{8Q_j} + H_{8Q_j}\chi_{8Q_j}$ a.e. and χ_{8Q_j} is the characteristic function of $8Q_j$. Let c_p be the weak-type (p, p) bound of the maximal function $M_{\rho,\theta}$. From (3.2) and $\bar{x}_j \in Q_j \subset 8Q_j$, we obtain

$$\begin{aligned} |\{M_{\rho,\theta}(G\chi_{8Q_j}) > (2K/\tilde{C}_0)\lambda\}| &\leq \frac{2\tilde{C}_0c_1}{K\lambda} \int_{8Q_j} G_{8Q_j}(y) dy \\ &\leq \frac{2C_0c_1}{K\lambda} |8Q_j| G(\bar{x}_j) \leq \frac{8^n\tilde{C}_0c_1}{K\lambda} |Q_j| \gamma. \end{aligned}$$

Next assume first that $q < \infty$. By (3.1) and $x_j, \bar{x}_j \in 8Q_j$, we obtain

$$\begin{aligned} |\{M_{\rho,\theta}(H\chi_{8Q_j}) > (2K/\tilde{C}_0)\lambda\}| &\leq \left(\frac{2\tilde{C}_0c_q}{K\lambda} \right)^q \int_{8Q_j} H_{8Q_j}^q(y) dy \\ &\leq \left(\frac{2\tilde{C}_0c_q}{K\lambda} \right)^q |8Q_j| a^q (M_{\rho,\theta}F(x_j) + G(\bar{x}_j))^q \\ &\leq \left(\frac{4\tilde{C}_0c_q a 8^n}{K} \right)^q |Q_j|. \end{aligned}$$

From the two inequalities above, we get

$$|B_\lambda \cap Q_j| \leq C \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right) |Q_j|.$$

Note that $\omega \in RH_{s'}^{loc}$. If $s' < \infty$, for any cube $Q = Q(x_0, r)$ with $r < \rho(x_0)$ and any measurable set $E \subset Q$ we have

$$\frac{\omega(E)}{\omega(Q)} \leq \frac{|Q|}{\omega(Q)} \left(\frac{1}{|Q|} \int_Q w(y)^{s'} dy \right)^{1/s'} \left(\frac{|E|}{|Q|} \right)^{1/s} \leq C_\omega \left(\frac{|E|}{|Q|} \right)^{1/s}.$$

Note that the same conclusion holds in the case $s' = \infty$. Applying this to $B_\lambda \cap Q_j \subset Q_j$, we have

$$\omega(B_\lambda \cap Q_j) \leq C_\omega C \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{1/s} \omega(Q_j).$$

Since the Whitney cubes are disjoint we get

$$\omega(B_\lambda) \leq \sum_j \omega(B_\lambda \cap Q_j) \leq C \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{1/s'} \sum_j \omega(Q_j) = C \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{1/s} \omega(E_\lambda)$$

which is (3.3).

When $q = \infty$, then by (3.1)

$$\|M_{\rho,\theta}(H_{8Q_j\chi_{8Q_j}})\|_{L^\infty} \leq \|H_{8Q_j\chi_{8Q_j}}\|_{L^\infty} \leq a(M_{\rho,\theta}F(x_j) + G(\bar{x}_j)) \leq 2a\lambda.$$

Thus, choosing $K \geq 4a\tilde{C}_0$ it follows that $\{M_{\rho,\theta}(H_{8Q_j\chi_{8Q_j}}) > (K/2\tilde{C}_0)\lambda\} = \emptyset$. Hence, we can obtain the desired result (with $K^{-q} = 0$).

When $M_{\rho,\theta}F \in L^p(\omega)$, we show (3.4). If $q < \infty$, integrating the two-parameter good- λ inequality (3.3) against $p\lambda^{p-1}d\lambda$ on $(0, \infty)$, for $0 < p < \infty$,

$$\|M_{\rho,\theta}F\|_{L^p(\omega)}^p \leq CK^p \left(\frac{a^q}{K^q} + \frac{\gamma}{K} \right)^{1/s} \|M_{\rho,\theta}F\|_{L^p(\omega)}^p + \frac{2^p K^p}{\gamma^p} \|G\|_{L^p(\omega)}^p.$$

For $0 < p < q/s$ we can choose K large enough and then γ small enough, we can get (3.4). In the same way, if $M_{\rho,\theta}F \in L^{p,\infty}(\omega)$, one shows the corresponding estimate in $L^{p,\infty}(\omega)$.

Observe that in the case $q = \infty$, K is already chosen and we only have to take some small γ . Thus, the corresponding estimates holds for $0 < p < \infty$ no matter the value of s .

Finally, we consider the case $p \geq 1$ and $F \in L^1$. By the standard method in pages 247-248 in [3], we can obtain the desired result. \square

Remark 3.1. In Theorem 3.1, if $q = \infty$, in fact, we only need $\omega \in A_{\rho,\infty}^\infty = \bigcup_{p \geq 1} A_p^{\rho,\infty}$, no matter the value of s . If $s > 1$ and $q < \infty$, then one also obtains the endpoint $p = q/s$. In addition, it should be worth pointing out that Theorem 3.1 generalizes Theorem 2.1 in [23].

Next we give an application of Theorem 3.1 toward weighted norm inequalities for operators, avoiding all use of kernel representation.

Theorem 3.2. *Let $1 \leq p_0 < q_0 \leq \infty$. Let T is sublinear operator acting on L^{p_0} . Let $\{A_r\}_{r>0}$ be family of operators acting from L_c^∞ into L^{p_0} . Assume that for any $\theta > 0$*

$$\begin{aligned} & \sup_{x \in B, r < \rho(x_0)} \frac{1}{|c_0 B|} \int_{c_0 B(x_0, r)} |T(I - A_{r(c_0 B)})f(y)|^{p_0} dy \\ & + \sup_{x \in B, r \geq \rho(x_0)} \frac{1}{\Psi_\theta(B)|B|} \int_{B(x_0, r)} |Tf(y)|^{p_0} dy \leq C_\theta M_{\rho,\theta}(|f|^{p_0})(x), \quad \forall x \in \mathbb{R}^n \end{aligned} \quad (3.5)$$

and

$$\left(\frac{1}{|c_0 B|} \int_{c_0 B} T(A_{r(c_0 B)})f(y)^{q_0} dy \right)^{1/q_0} \leq C_\theta M_{\rho,\theta}(|f|^{p_0})^{1/p_0}(x), \quad (3.6)$$

for any ball $B = B(x_0, r)$ with $r < \rho(x_0)$ and all $x \in c_0 B$ and $r(c_0 B)$ denotes $c_0 B$ radius and c_0 is a constant depending only on n . Let $p_0 < p < q_0$ (or $p = q_0$ when $q_0 < \infty$) and $\omega \in A_{p/p_0}^{\rho,\infty} \cap RH_{(q_0/p)}^{loc}$. There exists a constant C such that

$$\|Tf\|_{L^p(\omega)} \leq C \|Tf\|_{L^{p_0}(\omega)} \quad (3.7)$$

for all $f \in L_c^\infty(\mathbb{R}^n)$.

Proof. We first notice that Theorem 3.1 still holds if the cubes Q and $8Q$ in the conditions 3.1 and 3.2 are replaced by the balls B and c_0B respectively, where c_0 is a constant depending only on n . We now consider $q_0 < \infty$ and $p_0 < p \leq q_0$. Let $f \in L_c^\infty$ and so $F = |Tf|^{p_0} \in L^1$. Fix a ball $B = B(x_0, r)$ with $r < \rho(x_0)$. As T is sublinear, we have

$$F \leq G_{c_0B} + H_{c_0B} \equiv 2^{p_0-1}|T(I - A_{r(c_0B)})f|^{p_0} + 2^{p_0-1}|TA_{r(c_0B)}f|^{p_0}.$$

Then (3.5) and (3.6) yield the corresponding conditions (3.1) and (3.2) with $q = q_0/p_0$, $H_1 \equiv 0$, $a = 2^{p_0-1}C^{p_0}$ and $G = 2^{p_0-1}M_{\rho,\theta}(|f|^{p_0})$. Since $\omega \in RH_{(q_0/p)'}^{loc}$, then one pick $1 < s < q_0/p$ such that $\omega \in RH_{s'}^{loc}$. Thus, Lemma 2.3 and Theorem 3.1 with $p/p_0 > 1$ in place of p and $s = q_0/p$ yield

$$\begin{aligned} \|Tf\|_{L^p(\omega)}^{p_0} &\leq \|M_{\rho,\theta}F\|_{L^{p/p_0}(\omega)} \leq C\|G\|_{L^{p/p_0}(\omega)} \\ &= C\|M_{\rho,\theta}(|f|^{p_0})\|_{L^{p/p_0}(\omega)} \leq C\|f\|_{L^{p/p_0}(\omega)}, \end{aligned}$$

where in the last estimate we have used the fact that there exists $\theta_1 > 0$ such that $\omega \in A_{p/p_0}^{\rho,\theta_1}$ (since $\omega \in A_{p/p_0}^{\rho,\infty}$) and $\theta = \theta_1(p/p_0)'$.

In the case $q_0 = \infty$ and $p < \infty$, Theorem 3.1 applies as before when $\omega \in A_{p/p_0}^{\rho,\infty}$ by Remark 3.1. \square

A slight strengthening of the hypotheses in Theorem 3.1 furnishes weighted L^p estimates for commutators $BMO_\theta(\rho)$ functions. For any $k \in \mathbb{N}$ we define the k th order commutator

$$T_b^k f(x) = T((b(x) - b)^k f)(x), \quad f \in L_c^\infty, \quad x \in \mathbb{R}^n.$$

Note that $T_b^0 = T$. Commutators are usually considered for linear operators T in which case they can be alternatively defined by recurrence: the first order commutators is

$$T_b^1 f(x) = [b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

and for $k \geq 2$, the k th order commutators is given by $T_b^k = [b, T_b^{k-1}]$.

We claim that since T is bounded in L^{p_0} then $T_b^k f$ is well defined in L_{loc}^q for any $0 < q < p_0$ and for any $f \in L_c^\infty$: take a ball B containing the support of f and observe that by sublinearity for a.e. $x \in \mathbb{R}^n$

$$|T_b^k f(x)| \leq \sum_{m=0}^k C_{m,k} |b - b_B|^{k-m} |T((b - b_B)^m f)(x)|.$$

Lemma 2.4 implies

$$\int_B |b(y) - b_Q|^{mp_0} |f(y)|^{p_0} dy \leq C \|f\|_{L^\infty}^{p_0} \|b\|_{BMO_\theta}^{mp_0} |B| \Psi_{mp_0\theta'}(B) < \infty.$$

Hence, $T((b - b_B)^m f) \in L^{p_0}$ and the claim follows.

Theorem 3.3. *Let $1 \leq p_0 < q_0 \leq \infty$. Let T is sublinear operator acting on L^{p_0} . Let $\{A_r\}_{r>0}$ be family of operators acting from L_c^∞ into L^{p_0} . Assume that for any $\theta > 0$*

$$\begin{aligned} & \left(\frac{1}{|c_0 B|} \int_{c_0 B(x_0, r)} |T(I - A_{r(c_0 B)})f(y)|^{p_0} dy \right)^{1/p_0} \\ & \leq C_\theta \sum_{j=1}^{\infty} \alpha_j \left(\frac{1}{\Psi_\theta(B_{j+1})|B_{j+1}|} \int_{B_{j+1}} |f(y)|^{p_0} dy \right)^{1/p_0} \end{aligned} \quad (3.7)$$

holds for any ball $B = B(x_0, r)$ with $r < \rho(x_0)$ and $B_{j+1} = 2^{j+1}c_0 B$,

$$\left(\frac{1}{\Psi_\theta(B)|B|} \int_{B(x_0, r)} |Tf(y)|^{p_0} dy \right)^{\frac{1}{p_0}} \leq C_\theta \sum_{j=1}^{\infty} \alpha_j \left(\frac{1}{\Psi_\theta(B_{j+1})|B_{j+1}|} \int_{B_{j+1}} |f(y)|^{p_0} dy \right)^{\frac{1}{p_0}} \quad (3.8)$$

holds for any ball $B = B(x_0, r)$ with $r \geq \rho(x_0)$ and $B_{j+1} = 2^{j+1}B$, and

$$\left(\frac{1}{|c_0 B|} \int_{c_0 B} T(A_{r(c_0 B)})f(y)|^{q_0} dy \right)^{\frac{1}{q_0}} \leq C_\theta \sum_{j=1}^{\infty} \alpha_j \left(\frac{1}{\Psi_\theta(B_{j+1})|B_{j+1}|} \int_{B_{j+1}} |Tf(y)|^{p_0} dy \right)^{\frac{1}{p_0}}, \quad (3.9)$$

holds for any ball $B = B(x_0, r)$ with $r < \rho(x_0)$ and $B_{j+1} = 2^{j+1}c_0 B$. Let $p_0 < p < q_0$ (or $p = q_0$ when $q_0 < \infty$) and $\omega \in A_{p/p_0}^{\rho, \infty} \cap RH_{(q_0/p)'}^{loc}$. If $\sum_j \alpha_j^k < \infty$, then there exists a constant C such that for all $f \in L_c^\infty(\mathbb{R}^n)$ and $b \in BMO_{\theta_1}(\rho)$,

$$\|T_b^k f\|_{L^p(\omega)} \leq C \|b\|_{BMO_{\theta_1}(\rho)}^k \|Tf\|_{L^p(\omega)}. \quad (3.10)$$

Proof. We only prove the case $k = 1$, the case $k \geq 2$ can be deduced by induction. Let us fix $p_0 < p < q_0$ and $\omega \in A_{p/p_0}^{\rho, \theta_2} \cap RH_{(q_0/p)'}^{loc}$. We assume that $q_0 < \infty$, for $q_0 = \infty$ is similar. Without loss of generality, $b \in BMO_{\theta_1}(\rho) \cap L^\infty$ and $f \in L_c^\infty$, then $|T_b^1 f|^{p_0} \in L^1$. Set $F = |T_b^1 f|^{p_0}$. In the proof, we always assume $\eta > (l_0 + 1)(\theta_1 + \theta_2)$ large enough and $\eta_1 = p_0 2^{[p_0]+1} \eta$.

Given a ball $B = B(x_0, r)$, we first consider the case $r < \rho(x_0)$, we set $f_{\bar{B}, b} = (b_{4\bar{B}} - b)f$, $\bar{B} = c_0 B$ and decompose T_b^1 as follows:

$$\begin{aligned} |T_b^1 f(x)| &= |T((b(x) - b)f)(x)| \leq |b(x) - b_{\bar{B}}| |Tf(x)| + |T((b_{4\bar{B}} - b)f)(x)| \\ &\leq |b(x) - b_{\bar{B}}| |Tf(x)| + |T(I - A_{r(\bar{B})})f_{\bar{B}, b}(x)| + |TA_{r(\bar{B})}f_{\bar{B}, b}(x)|. \end{aligned}$$

We observe that $F \leq G_{\bar{B}} + H_{\bar{B}}$ where

$$G_{\bar{B}} = 4^{p_0} (G_{\bar{B}, 1} + G_{\bar{B}, 2}) = 4^{p_0-1} (|b - b_{4\bar{B}}|^{p_0} |Tf|^{p_0} + |T(I - A_{r(\bar{B})})f_{\bar{B}, b}|^{p_0})$$

and $H_{\bar{B}} = 2^{p_0-1} |TA_{r(\bar{B})}f_{\bar{B}, b}|^{p_0}$. Fix any $x \in \bar{B}$, by (2.1), we have

$$\frac{1}{|\bar{B}|} \int_{\bar{B}} G_{\bar{B}, 1}(y) dy = \frac{1}{|\bar{B}|} \int_{\bar{B}} |b - b_{4\bar{B}}|^{p_0} |Tf|^{p_0}(y) dy \leq C \|b\|_{BMO_{\theta_1}(\rho)} M_{\rho, \eta}^{[p_0]+2}(|Tf|^{p_0})(x).$$

Using (3.7), (2.1), and (2.3), and note that $\sum_i \alpha_j j < \infty$, we get

$$\begin{aligned}
\left(\frac{1}{|\bar{B}|} \int_{\bar{B}} G_{\bar{B},2}(y) dy \right)^{1/p_0} &= \left(\frac{1}{|\bar{B}|} \int_{\bar{B}} T(I - A_{r(\bar{B})}) f_{\bar{B},b} |^{p_0}(y) dy \right)^{1/p_0} \\
&\leq C \sum_{j=1}^{\infty} \alpha_j \left(\frac{1}{\Psi_{\eta_1}(B_{j+1})|B_{j+1}|} \int_{B_{j+1}} |f_{\bar{B},b}|^{p_0}(y) dy \right)^{1/p_0} \\
&\leq C \sum_{j=1}^{\infty} \alpha_j \|(b - b_{4\bar{B}})/\Psi_{\eta_1}(B_{j+1})\|_{\exp L, B_{j+1}} M_{\rho, \eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(x) \\
&\leq C \|b\|_{BMO_{\theta_1}(\rho)} M_{\rho, \eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(x) \sum_{j=1}^{\infty} \alpha_j j \\
&\leq C \|b\|_{BMO_{\theta_1}(\rho)} M_{\rho, \eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(x).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\left(\sup_{x \in \bar{B}, r < \rho(x_0)} \frac{1}{|\bar{B}|} \int_{\bar{B}} G_{\bar{B}}(y) dy \right)^{1/p_0} &\leq C(M_{\rho, \eta}^{[p_0]+2}(|Tf|^{p_0})^{\frac{1}{p_0}}(x) + M_{\rho, \eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(x)) \\
&\equiv G(x).
\end{aligned}$$

We next estimate the average of $H_{\bar{B}}^q$ on \bar{B} with $q = q_0/p_0$. Using (3.9) with $\theta = \eta_1$, we get

$$\begin{aligned}
\left(\frac{1}{|\bar{B}|} \int_{\bar{B}} H_{\bar{B}}^q(y) dy \right)^{1/q_0} &\leq C \left(\frac{1}{|\bar{B}|} \int_{\bar{B}} T A_{r(\bar{B})} f_{\bar{B},b} |^{p_0}(y) dy \right)^{1/p_0} \\
&\leq C \sum_{j=1}^{\infty} \alpha_j \left(\frac{1}{\Psi_{\eta_1}(B_{j+1})|B_{j+1}|} \int_{B_{j+1}} |T f_{\bar{B},b}|^{p_0}(y) dy \right)^{1/p_0} \\
&\leq C(M_{\rho, \eta_1} F)^{\frac{1}{p_0}}(x) \\
&\quad + C \sum_{j=1}^{\infty} \alpha_j \left(\frac{1}{\Psi_{\eta_1}(B_{j+1})|B_{j+1}|} \int_{B_{j+1}} |b - b_{4\bar{B}}|^{p_0} |Tf|^{p_0}(y) dy \right)^{1/p_0} \\
&\leq C(M_{\rho, \eta_1} F)^{\frac{1}{p_0}}(x) \\
&\quad + C \sum_{j=1}^{\infty} \alpha_j \|(b - b_{4\bar{B}})/\Psi_{\eta_1}(B_{j+1})\|_{\exp L, B_{j+1}} M_{\rho, \eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(\bar{x}) \\
&\leq C(M_{\rho, \eta_1} F)^{\frac{1}{p_0}}(x) + C \|b\|_{BMO_{\theta_1}(\rho)} M_{\rho, \eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(\bar{x}).
\end{aligned}$$

for any $x, \bar{x} \in \bar{B}$. Thus

$$\begin{aligned}
\left(\frac{1}{|\bar{B}|} \int_{\bar{B}} H_{\bar{B}}^q(y) dy \right)^{1/q_0} &\leq C(M_{\rho, \eta_1} F)^{\frac{1}{p_0}}(x) + M_{\rho, \eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(\bar{x}) \\
&\equiv C((M_{\rho, \eta_1} F)^{\frac{1}{p_0}}(x) + H_1(\bar{x})).
\end{aligned}$$

Given a ball $B = B(x_0, r)$, we first consider the case $r \geq \rho(x_0)$, we set $f_{B,b} = (b_{4B} - b)f$, $\bar{B} = B$ and decompose T_b^1 as follows:

$$|T_b^1 f(x)| = |T((b(x) - b)f)(x)| \leq |b(x) - b_B| |Tf(x)| + |T(f_{B,b})(x)|.$$

Fix any $x \in B$, by (2.1), we have

$$\frac{1}{|B|} \int_B |b - b_{4B}|^{p_0} |Tf|^{p_0}(y) dy \leq C \|b\|_{BMO_{\theta_1}(\rho)} M_{\rho,\eta}^{[p_0]+2}(|Tf|^{p_0})(x).$$

Using (3.8) with $\theta = \eta_1$, by (2.1) and (2.3), we get

$$\begin{aligned} & \left(\frac{1}{\Psi_{\eta_1}(B)|B|} \int_B |T(f_{B,b})|^{p_0}(y) dy \right)^{1/p_0} \\ & \leq C \sum_{j=1}^{\infty} \alpha_j j \left(\frac{1}{\Psi_{\eta_1}(B_{j+1})|B_{j+1}|} \int_{B_{j+1}} |f_{B,b}|^{p_0}(y) dy \right)^{1/p_0} \\ & \leq C \sum_{j=1}^{\infty} \alpha_j j \|(b - b_{4B})/\Psi_{\eta(B_{j+1})}\|_{\exp L, B_{j+1}} M_{\rho,\eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(x) \\ & \leq C \|b\|_{BMO_{\theta_1}} M_{\rho,\eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(x) \sum_{j=1}^{\infty} \alpha_j j \\ & \leq C \|b\|_{BMO_{\theta_1}} M_{\rho,\eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(x), \end{aligned}$$

Hence, we have

$$\begin{aligned} & \left(\sup_{x \in B, r \geq \rho(x_0)} \frac{1}{\Psi_{\eta_1}(B)|B|} \int_B F(y) dy \right)^{\frac{1}{p_0}} \leq C (M_{\rho,\eta}^{[p_0]+2}(|Tf|^{p_0})^{\frac{1}{p_0}}(x) + M_{\rho,\eta}^{[p_0]+2}(|f|^{p_0})^{\frac{1}{p_0}}(x)) \\ & \equiv G(x). \end{aligned}$$

Thus, applying Lemma 2.3 and Theorem 3.1, we get

$$\begin{aligned} \|T_b^1 f\|_{L^p(\omega)}^{p_0} & \leq \|M_{\rho,\eta_1} F\|_{L^{p/p_0}(\omega)} \leq C \|G\|_{L^{p/p_0}(\omega)} + C \|H_1\|_{L^{p/p_0}(\omega)} \\ & \leq C \|M_{\rho,\eta}^{[p_0]+2}(|f|^{p_0})\|_{L^{p/p_0}(\omega)} + C \|M_{\rho,\eta}^{[p_0]+2}(|Tf|^{p_0})\|_{L^{p/p_0}(\omega)} \\ & \leq C \|f\|_{L^p(\omega)}^{p_0} + \|Tf\|_{L^p(\omega)}^{p_0} \leq C \|f\|_{L^p(\omega)}^{p_0}, \end{aligned}$$

if η is large enough. □

4. Spectral multipliers

Suppose that L is a nonnegative self-adjoint operator acting on $L^2(\mathbb{R}^n)$. Let $E(\lambda)$ be the spectral resolution of L . By the spectral theorem, for any bounded Borel function $F : [0, \infty) \rightarrow \mathbb{C}$ one can define the operator

$$F(L) = \int_0^\infty F(\lambda) dE(\lambda),$$

which is bounded on $L^2(\mathbb{R}^n)$. The question of L^p estimates for functions of a self-adjoint operator is a delicate one. In fact, even for a Schrödinger operator $H = -\Delta + V(x)$ with a nonnegative potential, and a bound smooth kernel and hence does not fall within the scope of the Calderón-Zygmund theory. The first to overcome this difficulty was Hebisch [16]. Later, J. Dziubański [13] gave a spectral multiplier theorem for H^1 spaces associated with Schrödinger operators with potentials satisfying a reverse Hölder inequality. On the other hand, X. T. Doung, etc [11] showed that a sharp spectral multiplier for a non-negative self-adjoint operator L was obtained under the assumption of the kernel $p_t(x, y)$ of the analytic semigroup e^{-tL} having a Gaussian upper bound. Recently, T. Doung, etc [12] generalized the main results [11] to weighted cases; see also [7].

A natural problem considered in the spectral multiplier theory is to give sufficient conditions on F and L which imply the weighted boundedness of $F(L)$ associated with Schrödinger operators.

In this section, we always assume that L is a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ and that the semigroup e^{-tL} , generated by $-L$ on $L^2(\mathbb{R}^n)$, has the kernel $p_t(x, y)$ which satisfies the following Gaussian upper bound

$$|p_t(x, y)| \leq C_N t^{-n/2} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)} \right)^{-N} \exp \left(-b \frac{|x - y|^2}{t} \right) \quad (4.1)$$

for all $t > 0$, $N > 0$, and $x, y \in \mathbb{R}^n$, where the auxiliary function $\rho(x)$ satisfies Lemma 2.1, C_N depends only N , and b is a positive constant.

Such estimates are typical for divergence Schrödinger differential operator

$$L = -\partial_i(a_{ij}(x)\partial_j) + V(x) \text{ on } \mathbb{R}^n, \quad n \geq 3,$$

where $V(x) \in RH_{n/2}$ is a nonnegative potential, and a_{ij} satisfy (1.1) (see [13]).

Suppose that T is a bounded operator on L^2 . We say that a measurable function $K_T : \mathbb{R}^{2n} \rightarrow \mathbb{C}$ is the (singular) kernel of T if

$$\langle T f_1, f_2 \rangle = \int_{\mathbb{R}^n} T f_1 \bar{f}_2 dx = \int_{\mathbb{R}^n} K_T(x, y) f_1(y) \overline{f_2(x)} dx dy$$

for all $f_1, f_2 \in C_c(\mathbb{R}^n)$ (for all $f_1, f_2 \in C_c(\mathbb{R}^n)$ such that $\text{supp} f_1 \cap \text{supp} f_2 = \emptyset$, respectively).

Theorem 4.1. *Let T be a non-negative self-adjoint operator such that the corresponding heat kernel satisfy (4.1). Suppose that $F[0, \infty) \rightarrow \mathbb{C}$ is a bounded Borel function such that*

$$\sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty} \leq C_s < \infty, \quad (4.2)$$

for any $s > 0$, where $\delta_t F(\lambda) = F(t\lambda)$, $\|F\|_{W_s^\infty} = \|(I - d^2/d^2)^{s/2} F\|_{L^\infty}$ and $\eta \in C_c^\infty(\mathbb{R}_+)$ is a fixed function, non identically zero. Then the operator $F(L)$ is bounded on $L^p(\omega)$ for all p and ω satisfying $1 < p < \infty$ and $\omega \in A_p^{\rho, \infty}$, and is of weighted weak type $(1, 1)$ for weights $\omega \in A_1^{\rho, \infty}$. In addition,

$$\|F(L)f\|_{L^p(\omega) \rightarrow L^p(\omega)} \leq C_s \left(\sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty} + |F(0)| \right), \quad (4.3)$$

for $1 < p < \infty$, $\omega \in A_p^{\rho, \infty}$ provided that s is large enough, and

$$\|F(L)f\|_{L^p(\omega) \rightarrow L^p(\omega)} \leq C_s \left(\sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty} + |F(0)| \right), \quad (4.4)$$

for $\omega \in A_1^{\rho, \infty} \cap RH_2^{loc}$ provided that s is large enough.

For the commutators for $F(L)$, we have the following results.

Theorem 4.2. *Let T be a non-negative self-adjoint operator such that the corresponding heat kernel satisfy (4.1). Suppose that $F[0, \infty) \rightarrow \mathbb{C}$ is a bounded Borel function such that*

$$\sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty} \leq C_s < \infty,$$

for any $s > 0$, where $\delta_t F(\lambda) = F(t\lambda)$, $\|F\|_{W_s^\infty} = \|(I - d^2/d^2)^{s/2} F\|_{L^\infty}$ and $\eta \in C_c^\infty(\mathbb{R}_+)$ is a fixed function, non identically zero. Let $b \in BMO_\theta(\rho)$ and $k \in \mathbb{N}$, then for k -order commutator $F_b^k(L)$, we have

$$\|F_b^k(L)f\|_{L^p(\omega)} \leq C \|b\|_{BMO_\theta(\rho)}^k \|f\|_{L^p(\omega)}, \quad (4.5)$$

for $1 < p < \infty$, $\omega \in A_p^{\rho, \infty}$, and

$$\omega(\{x \in \mathbb{R}^n : |F_b^k(L)f(x)| > \lambda\}) \leq C \Phi(\|b\|_{BMO_\theta(\rho)}) \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \omega(x) dx, \quad (4.6)$$

where $\Phi(t) = t \log(e + t)^k$ and $\omega \in A_1^{\rho, \infty} \cap RH_2^{loc}$.

Remark 4.1. Let $F(\lambda) = \lambda^{i\gamma}$ with $\gamma \in \mathbb{R}$, $F(\lambda) = \cos \lambda$, $F(\lambda) = \sin \lambda$, or $F(\lambda) \in C_0^\infty(0, \infty)$, then these F all satisfy (4.2).

Remark 4.2. In fact, Theorems 3.1, 3.2, 3.3 and Theorems 4.1 and 4.2 still hold on RD spaces; see [25] for more details.

4.1. Some lemmas

To prove Theorems 4.1 and 4.2, we need some lemmas.

Lemma 4.1. *Suppose that (4.1) holds. Then for any $N > 0$ such that*

$$\int_{\mathbb{R}^n \setminus B(y, r)} |p_t(x, y)|^2 dx \leq C t^{-\frac{n}{2}} \left(1 + \frac{\sqrt{t}}{\rho(y)}\right)^{-N} \exp\left(-b \frac{r^2}{t}\right)$$

The proof is obviously by (4.1).

Lemma 4.2. *Suppose that (4.1) holds. For any $s \geq 0$ and $N > 0$ there exists a constant C such that*

$$\int_{\mathbb{R}^n} |p_{(1+i\tau)R^{-2}}(x, y)|^2 |x - y|^s dx \leq C R^{n-s} (1 + |\tau|)^s \left(1 + \frac{R}{\rho^2(y)}\right)^{-N}. \quad (4.7)$$

where $p_{(1+i\tau)R^{-2}} = K_{\exp((1+i\tau)R^{-2}L)}$.

Proof. Assume $\|f\|_{L^2}$ and $f \in \mathbb{R}^n \setminus B(y, r)$. we defines the holomorphic function $F_y : \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \rightarrow \mathbb{C}$ by the formula

$$F_y(z) = e^{-zR^2} R^n \left(1 + \frac{z}{\rho^2(y)}\right)^{-N} \left(\int_{\mathbb{R}^n} p_z(x, y) f(x) dx\right)^2.$$

If let $z = |z|e^{i\theta}$, then

$$\|p_z(\cdot, y)\|_{L^2}^2 = \|p_{|z|\cos\theta}(\cdot, y)\|_{L^2}^2.$$

From this and Lemma 4.1, we get

$$\begin{aligned} |F_y(z)| &\leq e^{-R^2|z|\cos\theta} R^{-n} \|p_{|z|\cos\theta}(\cdot, y)\|_{L^2}^2 \\ &\leq e^{-R^2|z|\cos\theta} R^{-n} (|z|\cos\theta)^{-\frac{1}{2}} \\ &\leq CR^{-n} (|z|\cos\theta)^{-\frac{n}{2}}. \end{aligned}$$

Similarly, for $\theta = 0$ by Lemma 4.1,

$$|F_y(|z|)| \leq CR^n |z|^{-n/2} e^{-br^2/|z|}.$$

Combining the inequalities above and Lemma 9 in [9](see also the proof of Lemma 4.1 in [11]), we have

$$\begin{aligned} |F_y((1+i\tau)R^{-2})| &\leq Ce^{-b(rR/(1+|\tau|))^2}. \\ \int_{\mathbb{R}^n} |p_{(1+i\tau)R^{-2}}(x, y)|^2 dx &\leq CR^n \left(1 + \frac{1}{R\rho(y)}\right)^{-N} e^{-b(rR/(1+|\tau|))^2}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} |p_{(1+i\tau)R^{-2}}(x, y)|^2 |x - y|^s dx \\ &= \sum_{k \geq 0} \int_{k(1+|\tau|)R^{-1} \leq |x-y| \leq (k+1)(1+|\tau|)R^{-1}} |p_{(1+i\tau)R^{-2}}(x, y)|^2 |x - y|^s dx \\ &\leq (1+|\tau|)^s R^{-s} \sum_{k \geq 0} (k+1)^s \int_{\mathbb{R}^n \setminus B(y, (k+1)(1+|\tau|)R^{-1})} |p_{(1+i\tau)R^{-2}}(x, y)|^2 dx \\ &\leq CR^{n-s} (1+|\tau|)^s \left(1 + \frac{1}{R\rho(y)}\right)^{-N}. \end{aligned}$$

Thus, (4.7) holds. \square

Applying Lemmas 4.1 and 4.2, and adapting the similar arguments in the proof of lemma 4.2 in [11], we can obtain the following result.

Lemma 4.3. *Let $R > 0$, $s > 0$. Then for any $\epsilon > 0$ and $N > 0$, there exists a constant $C = C(s, \epsilon, N)$ such that*

$$\int_{\mathbb{R}^n} |K_{F(\sqrt{L})}|^2 (1 + R|x - y|)^s dx \leq CR^n \left(1 + \frac{1}{R\rho(y)}\right)^{-N} \|\delta_R F\|_{W_{s/2+\epsilon}^\infty}^2$$

for all Borel functions F such that $\operatorname{supp} F \subset [R/4, R]$.

Lemma 4.4. Suppose $\omega \in A_1^{\rho, \theta} \cap RH_2^{loc}$. Let $\eta = 3\theta + (l_0 + 2)n$, then for any $s > n + 2\eta$ such that

$$\int_{\mathbb{R}^n \setminus B(y, r)} (1 + R|x - y|)^{-s} \omega(x)^2 dx \leq CR^{-n} (1 + rR)^{n+2\eta-s} \left(1 + \frac{1}{r\rho(y)}\right)^{-2\eta} \omega(y)^2.$$

Proof. Let $\eta_1 = 2\theta + (l_0 + 1)n$. Assume that $rR > 1$. Then

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus B(y, r)} (1 + R|x - y|)^{-s} \omega(x)^2 dx \\ & \leq \sum_{k \geq 0} \int_{2^k r \leq |x-y| \leq 2^{k+1} r} (R|x - y|)^{-s} \omega(x)^2 dx \\ & \leq \sum_{k \geq 0} (2^k r R)^{-s+n} \frac{1}{(2^{k+1} r)^n} \int_{|x-y| \leq 2^{k+1} r} \omega(x)^2 dx \\ & \leq C \sum_{k \geq 0} (2^k r R)^{-s+n} \left(\frac{1}{(2^{k+1} r)^n} \int_{|x-y| \leq 2^{k+1} r} \omega(x) dx \right)^2 \left(1 + \frac{2^{k+1} r}{\rho(y)} \right)^{2\eta_1} \\ & \leq C \sum_{k \geq 0} (2^k r R)^{-s+n} (M_{\rho, \theta} w(y))^2 \left(1 + \frac{2^{k+1} r}{\rho(y)} \right)^{2\eta} \\ & \leq C \sum_{k \geq 0} (2^k r R)^{-s+n} w(y)^2 \left(1 + \frac{2^{k+1} r R}{R\rho(y)} \right)^{2\eta} \\ & \leq C \sum_{k \geq 0} (2^k r R)^{-s+n+2\eta} w(y)^2 \left(1 + \frac{1}{R\rho(y)} \right)^{2\eta} \\ & \leq C (rR)^{-s+n+2\eta} w(y)^2 \left(1 + \frac{1}{R\rho(y)} \right)^{2\eta}, \end{aligned}$$

since $s > n + 2\eta$.

If $rR < 1$, note that $s > n + 2\eta$, we then have

$$\begin{aligned} & \int_{\mathbb{R}^n} (1 + R|x - y|)^{-s} \omega(x)^2 dx \leq \int_{|x-y| < 1/R} \omega(x)^2 dx \\ & \quad + \sum_{k \geq 1} 2^{-ks} \int_{2^k/R \leq |x-y| \leq 2^{k+1}/R} \omega(x)^2 dx \\ & \leq CR^{-n} \sum_{k \geq 0} 2^{k(-s+n+2\eta)} w(y)^2 \left(1 + \frac{1}{R\rho(y)} \right)^{2\eta} \\ & \leq CR^{-n} w(y)^2 \left(1 + \frac{1}{R\rho(y)} \right)^{2\eta}. \end{aligned}$$

□

Applying Lemmas 4.3 and 4.4, and adapting the similar arguments in the proof of Theorem 3.1 in [11], we can obtain the following result.

Lemma 4.5. Suppose $\omega \in A_1^{\rho, \theta} \cap RH_2^{loc}$ and $\eta = 3\theta + (l_0 + 2)n$. If for any $s > n/2 + \eta$ such that $\sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty} \leq C_s$, then

$$\int_{\mathbb{R}^n \setminus B(y, r)} |K_{F(1-\Phi_r)(\sqrt{L})}(x, y)| \omega(x) dx \leq C\omega(y), \quad \text{a.e. } y \in \mathbb{R}^n, \quad (4.8)$$

where $\Phi_r(\lambda) = \exp(-(\lambda r)^2)$.

Lemma 4.6. Suppose $\omega \in A_1^{\rho, \theta} \cap RH_2^{loc}$ and $b \in BMO_{\theta_1}(\rho)$. Let $\eta = (l_0 + 1)\theta_1 + 3\theta + (l_0 + 2)n$, then there exists $s > n + 2\eta$ such that

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus B(y, r)} |b(x) - b_B|^2 (1 + R|x - y|)^{-s} \omega(x)^2 dx \\ & \leq C \|b\|_{BMO_{\theta_1}(\rho)}^2 R^{-n} (1 + rR)^{n+2\eta-s} \left(1 + \frac{1}{r\rho(y)}\right)^{-2\eta} \omega(y)^2, \end{aligned}$$

where $y \in B = B(x_0, r)$ with $r < \rho(x_0)$ and b_B is the average on B .

Proof. Let $\eta_1 = 2\theta + (l_0 + 1)(n + \theta_1)$. Assume that $rR > 1$. Since $\omega \in RH_2^{loc}$, So $\omega \in RH_{2\gamma}^{loc}$ for some $\gamma > 1$. Let $1/\gamma + 1/\gamma' = 1$. Then

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus B(y, r)} |b(x) - b_B|^2 (1 + R|x - y|)^{-s} \omega(x)^2 dx \\ & \leq \sum_{k \geq 0} \int_{2^k r \leq |x-y| \leq 2^{k+1} r} |b(x) - b_B|^2 (R|x - y|)^{-s} \omega(x)^2 dx \\ & \leq \sum_{k \geq 0} (2^k r R)^{-s+n} \frac{1}{(2^{k+1} r)^n} \int_{|x-y| \leq 2^{k+1} r} |b(x) - b_B|^2 \omega(x)^2 dx \\ & \leq \sum_{k \geq 0} (2^k r R)^{-s+n} \left(\frac{1}{(2^{k+1} r)^n} \int_{|x-y| \leq 2^{k+1} r} |b(x) - b_B|^{2\gamma'} dx \right)^{1/\gamma'} \\ & \quad \times \left(\frac{1}{(2^{k+1} r)^n} \int_{|x-y| \leq 2^{k+1} r} (\omega(x))^{2\gamma} dx \right)^{1/\gamma} \\ & \leq C \|b\|_{BMO_{\theta_1}(\rho)}^2 \sum_{k \geq 0} (2^k r R)^{-s+n} \left(\frac{1}{(2^{k+1} r)^n} \int_{|x-y| \leq 2^{k+1} r} \omega(x) dx \right)^2 \left(1 + \frac{2^{k+1} r}{\rho(y)} \right)^{2\eta_1} \\ & \leq C \|b\|_{BMO_{\theta_1}(\rho)}^2 \sum_{k \geq 0} (2^k r R)^{-s+n} (M_{\rho, \theta} w(y))^2 \left(1 + \frac{2^{k+1} r}{\rho(y)} \right)^{2\eta} \\ & \leq C \|b\|_{BMO_{\theta_1}(\rho)}^2 \sum_{k \geq 0} (2^k r R)^{-s+n} w(y)^2 \left(1 + \frac{2^{k+1} r R}{R\rho(y)} \right)^{2\eta} \\ & \leq C \|b\|_{BMO_{\theta_1}(\rho)}^2 \sum_{k \geq 0} (2^k r R)^{-s+n+2\eta} w(y)^2 \left(1 + \frac{1}{R\rho(y)} \right)^{2\eta} \\ & \leq C \|b\|_{BMO_{\theta_1}(\rho)}^2 (rR)^{-s+n+2\eta} w(y)^2 \left(1 + \frac{1}{R\rho(y)} \right)^{2\eta}. \end{aligned}$$

If $rR < 1$, similarly, we have

$$\begin{aligned}
& \int_{\mathbb{R}^n} |b(x) - b_B|^2 (1 + R|x - y|)^{-s} \omega(x)^2 dx \\
& \leq \int_{|x-y| < 1/R} |b(x) - b_B|^2 \omega(x)^2 dx \\
& \quad + \sum_{k \geq 1} 2^{-ks} \int_{2^k/R \leq |x-y| \leq 2^{k+1}/R} |b(x) - b_B|^2 \omega(x)^2 dx \\
& \leq C \|b\|_{BMO_{\theta_1}(\rho)}^2 R^{-n} \sum_{k \geq 0} 2^{k(-s+n+2\eta)} w(y)^2 \left(1 + \frac{1}{R\rho(y)}\right)^{2\eta} \\
& \leq C \|b\|_{BMO_{\theta_1}(\rho)}^2 R^{-n} w(y)^2 \left(1 + \frac{1}{R\rho(y)}\right)^{2\eta}.
\end{aligned}$$

□

Applying Lemmas 4.3 and 4.6, and adapting the similar arguments in the proof of Theorem 3.1 in [11], we can obtain the following result.

Lemma 4.7. *Suppose $\omega \in A_1^{\rho, \theta} \cap RH_2^{loc}$, $b \in BMO_{\theta_1}(\rho)$ and $\eta = (l_0 + 1)\theta_1 + 3\theta + (l_0 + 2)n$. If for any $s > n/2 + \eta$ such that $\sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty} \leq C_s$, then*

$$\int_{\mathbb{R}^n \setminus B(y, r)} |K_{F(1-\Phi_r)(\sqrt{L})}(x, y)| |b(x) - b_B| \omega(x) dx \leq C \|b\|_{BMO_{\theta_1}(\rho)} \omega(y),$$

where $\Phi_r(\lambda) = \exp(-(\lambda r)^2)$, $y \in B = B(x_0, r)$ with $r \leq \rho(x_0)$ and b_B is the average on B .

Lemma 4.8([23]). *For any a ball $B = B(x_0, r)$, if $r \geq \rho(x_0)$, then the ball B can be decomposed into finite disjoint cubes $\{Q_i\}_{i=1, m}$ such that $B \subset \bigcup_i^m Q_i \subset 2\sqrt{n}B$ and $r_i/2 \leq \rho(x) \leq 2\sqrt{n}C_0 r_i$ for some $x \in Q_i = Q(x_i, r_i)$, where C_0 is same as Lemma 2.1.*

4.2. Proof of Theorem 4.1

In this section, we borrow some ideas from [12]. We first prove (4.5). Since $\omega \in A_p^{\rho, \infty}$, then there exist $p_0 > 1$ and $\theta > 0$ such that $\omega \in A_{p/p_0}^{\rho, \theta}$ by Proposition (iii). We will show that for any $\eta > 0$

$$\begin{aligned}
& \sup_{x \in B, r < \rho(x_0)} \frac{1}{|c_0 B|} \int_{c_0 B(x_0, r)} |T(I - A_{r(c_0 B)})f(y)|^{p_0} dy \\
& + \sup_{x \in B, r \geq \rho(x_0)} \frac{1}{\Psi_\eta(B)|B|} \int_{B(x_0, r)} |Tf(y)|^{p_0} dy \leq CM_{\rho, \eta}(|f|^{p_0})(x), \quad \forall x \in \mathbb{R}^n
\end{aligned} \tag{4.9}$$

for all $f \in L_c^\infty(\mathbb{R}^n)$.

Let us now prove (4.9). Observe that $\sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty} \sim \sup_{t>0} \|\eta \delta_t G\|_{W_s^\infty}$ where $G(\lambda) = F(\sqrt{\lambda})$. So, we can replace $F(L)$ by $F(\sqrt{L})$ in the proof. Notice that $F(\lambda) = F(\lambda) - F(0) + F(0)$ and hence

$$F(\sqrt{L}) = (F(\cdot) - F(0))(\sqrt{L}) + F(0)I.$$

Replacing F by $F - F(0)$, we may assume in the sequel that $F(0) = 0$. Let $\varphi \in C_c^\infty(0, \infty)$ be a non-negative function satisfying $\text{supp } \varphi \subset [\frac{1}{4}, 1]$ and $\sum_{l=-\infty}^{\infty} \varphi(2^{-l}\lambda) = 1$ for any $\lambda > 0$, and let φ_l denote the function $\varphi(2^{-l}\cdot)$. Then

$$F(\lambda) = \sum_{l=-\infty}^{\infty} \varphi(2^{-l}\lambda)F(\lambda) = \sum_{l=-\infty}^{\infty} F^l(\lambda), \quad \forall \lambda \geq 0. \quad (4.10)$$

This decomposition implies that the sequence $\sum_{l=-N}^N F^l(\sqrt{\lambda})$ converges strongly in $L^2(\mathbb{R}^n)$ to $F(\sqrt{\lambda})$.

We first consider the case $B = B(x_0, r)$ with $r < \rho(x_0)$. For every $l \in \mathbb{Z}, r > 0$, $M \in \mathbb{N}$ and $\lambda > 0$, we set

$$F_{r,M}(\lambda) = F(\lambda)(1 - e^{-(r\lambda)^M})^M, \quad (4.11)$$

$$F_{r,M}^l(\lambda) = F^l(\lambda)(1 - e^{-(r\lambda)^M})^M, \quad (4.12)$$

We use the decomposition $f = \sum_{j=0}^{\infty} f_j$ in which $f_j = f\chi_{U_j(\bar{B})}$, where $U_0(B) = c_0B := \bar{B}$ and $U_j(\bar{B}) = 2^j\bar{B} \setminus 2^{j-1}\bar{B}$ for $j = 1, 2, \dots$. We set $r_B = c_0r$, then

$$\begin{aligned} F(\sqrt{L})(1 - e^{-r_B^2 L})^M f &= F_{r_B,M}(\sqrt{L})f \\ &= \sum_{j=1}^2 F_{r_B,M}(\sqrt{L})f_j + \lim_{N \rightarrow \infty} \sum_{l=-N}^N \sum_{j=3}^{\infty} F_{r_B,M}^l(\sqrt{L})f_j, \end{aligned} \quad (4.13)$$

where the sequence converges strongly in $L^2(\mathbb{R}^n)$.

Note that $\|e^{-tL}f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ for any $t > 0$, and the L^p -boundedness of the operator $F(\sqrt{L})$ (see Theorem 3.1 in [11]), for any $x \in B$, we have

$$\begin{aligned} \left(\frac{1}{|c_0B|} \int_{c_0B(x_0,r)} |F_{r_B,M}(\sqrt{L})f_j|^{p_0} dy \right)^{1/p_0} &\leq C|B|^{-1/p_0} \|F_{r_B,M}(\sqrt{L})f_j\|_{L^{p_0}(\mathbb{R}^n)} \\ &\leq CM_{\rho,\eta}(|f|^{p_0})^{1/p_0}(x), \end{aligned} \quad (4.14)$$

for $j = 1, 2$.

Fix $j \geq 3$. Let $p_1 \geq 2$ and $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{2}$. Adapting the same arguments in pages 1117-1119 in [12], using Lemma 4.3 with $N = 0$, we have

$$\begin{aligned} &\left(\frac{1}{|c_0B|} \int_{c_0B(x_0,r)} |F_{r_B,M}^l(\sqrt{L})f_j|^{p_0} dy \right)^{1/p_0} \\ &\leq C|B|^{-1/p_1} \|F_{r_B,M}^l(\sqrt{L})f_j\|_{L^{p_1}(\mathbb{R}^n)} \\ &\leq C|B|^{-1/p_1} \|F_{r_B,M}^l(\sqrt{L})\|_{L^{p_0}(U_j(\bar{B})) \rightarrow L^{p_1}(\bar{B})} \|f_j\|_{L^{p_0}(\mathbb{R}^n)} \\ &\leq C2^{j(\eta+n)/p_0} |B|^{\frac{1}{2}} \|F_{r_B,M}^l(\sqrt{L})\|_{L^{p_0}(U_j(\bar{B})) \rightarrow L^{p_1}(\bar{B})} M_{\rho,\eta}(|f|^{p_0})^{1/p_0}(x) \\ &\leq C2^{-js+j(\eta+n)/p_0} \left(\min\{1, (2^l r_B)^{2M}\} \max\{1, (2^l r_B)^{\frac{n}{2}}\} \right) \\ &\quad \times M_{\rho,\eta}(|f|^{p_0})^{1/p_0}(x) \sup_{l \in \mathbb{Z}} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty}. \end{aligned} \quad (4.15)$$

Hence,

$$\begin{aligned}
& \sum_{j=3}^{\infty} \sum_{l=-\infty}^{\infty} \left(\frac{1}{|c_0 B|} \int_{c_0 B(x_0, r)} |F_{r_B, M}^l(\sqrt{L}) f_j|^{p_0} dy \right)^{1/p_0} \\
& \leq C \sum_{j=3}^{\infty} 2^{-js+j(\eta+n)/p_0} \left(\sum_{l=-\infty}^{\infty} \min\{1, (2^l r_B)^{2M}\} \max\{1, (2^l r_B)^{\frac{n}{2}}\} \right) \\
& \quad \times M_{\rho, \eta}(|f|^{p_0})^{1/p_0}(x) \sup_{l \in \mathbb{Z}} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty} \\
& \leq C \sum_{j=3}^{\infty} 2^{-js+j(\eta+n)/p_0} \left(\sum_{l: 2^l r_B > 1} (2^l r_B)^{-s+\frac{n}{2}} + \sum_{l: 2^l r_B \leq 1} (2^l r_B)^{2M-s} \right) \\
& \quad \times M_{\rho, \eta}(|f|^{p_0})^{1/p_0}(x) \sup_{l \in \mathbb{Z}} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty} \\
& \leq C M_{\rho, \eta}(|f|^{p_0})^{1/p_0}(x) \sup_{l \in \mathbb{Z}} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty},
\end{aligned} \tag{4.15}$$

if $s > n + \eta$ and $M > s/2$.

We now consider the case $B = B(x_0, r)$ with $r \geq \rho(x_0)$. Let $f = \sum_{j=0}^{\infty} f_j$ in which $f_j = f \chi_{U_j(B)}$ where $U_0(B) = B$ and $U_j(B) = 2^j B \setminus 2^{j-1} B$ for $j = 1, 2, \dots$. We write

$$F(\sqrt{L})f = \sum_{j=1}^2 F(\sqrt{L})f_j + \lim_{N \rightarrow \infty} \sum_{l=-N}^N \sum_{j=3}^{\infty} F^l(\sqrt{L})f_j,$$

where the sequence converges strongly in $L^2(\mathbb{R}^n)$.

Similar to the proof of (4.14), we have

$$\begin{aligned}
& \left(\frac{1}{|\Psi_\eta(B)| |B|} \int_{B(x_0, r)} |F(\sqrt{L})f_j|^{p_0} dy \right)^{1/p_0} \leq C (\Psi_\eta(B)|B|)^{-1/p_0} \|F(\sqrt{L})f_j\|_{L^{p_0}(\mathbb{R}^n)} \\
& \leq C M_{\rho, \eta}(|f|^{p_0})^{1/p_0}(x),
\end{aligned}$$

for $j = 1, 2$. Fix $j \geq 3$. Let $p_1 \geq 2$ and $\frac{1}{p_0} - \frac{1}{p_1} = \frac{1}{2}$. By Hölder inequality, for any $x \in B$, we have

$$\begin{aligned}
& \left(\frac{1}{|\Psi_\eta(B)| |B|} \int_{B(x_0, r)} |F(\sqrt{L})f_j|^{p_0} dy \right)^{1/p_0} \\
& \leq C |B|^{-1/p_1} \Psi_\eta(B)^{-1/p_0} \|F^l(\sqrt{L})f_j\|_{L^{p_1}(\mathbb{R}^n)} \\
& \leq C |B|^{-1/p_1} \|F^l(\sqrt{L})\|_{L^{p_0}(U_j(B)) \rightarrow L^{p_1}(B)} \Psi_\eta(B)^{-1/p_0} \|f_j\|_{L^{p_0}(\mathbb{R}^n)} \\
& \leq C 2^{j(\eta+n)/p_0} |B|^{\frac{1}{2}} \|F^l(\sqrt{L})\|_{L^{p_0}(U_j(B)) \rightarrow L^{p_1}(B)} M_{\rho, \eta}(|f|^{p_0})^{1/p_0}(x).
\end{aligned}$$

Let $\frac{1}{p_0} = \frac{\theta}{1} + \frac{1-\theta}{2}$ and $\frac{1}{p_1} = \frac{\theta}{2}$, that is $\theta = 2(\frac{1}{p_0} - \frac{1}{2})$. By interpolation,

$$\begin{aligned} & \|F^l(\sqrt{L})\|_{L^{p_0}(U_j(B)) \rightarrow L^{p_1}(B)} \\ & \leq \|F^l(\sqrt{L})\|_{L^2(U_j(B)) \rightarrow L^\infty(B)}^{1-\theta} \|\bar{F}^l(\sqrt{L})\|_{L^2(U_j(B)) \rightarrow L^\infty(B)}^\theta, \end{aligned}$$

where $\bar{F}(L)$ be the operator with multiplier \bar{F} , the complex conjugate of F , and \bar{F} satisfies the same estimates as F .

Next we estimate $\|F^l(\sqrt{L})\|_{L^2(U_j(B)) \rightarrow L^\infty(B)}$. For every $l \in \mathbb{Z}$, let $K_{F^l}(\sqrt{L})(y, z)$ be the Schwartz kernel of operator $F^l(\sqrt{L})$. Then

$$\begin{aligned} & \|F^l(\sqrt{L})\|_{L^2(U_j(B)) \rightarrow L^\infty(B)}^2 \\ & = \sup_{y \in B} \int_{U_j(B)} |K_{F^l}(\sqrt{L})(y, z)|^2 dz \\ & \leq C 2^{-2sj} (2^l r)^{-2s} \sup_{y \in B} \int_{\mathbb{R}^n} |K_{F^l}(\sqrt{L})(y, z)|^2 (1 + 2^l |y - z|)^{2s} dz. \end{aligned} \tag{4.16}$$

Applying Lemma 4.3 with $F = F^l$ and $R = 2^l$, we then have

$$\begin{aligned} & \int_{\mathbb{R}^n} |K_{F^l}(\sqrt{L})(y, z)|^2 (1 + 2^l |y - z|)^{2s} dz \leq C 2^{ln} \left(1 + \frac{1}{2^l \rho(y)}\right)^{-N} \|\delta_{2^l}(F^l)\|_{W_s^\infty}^2 \\ & = C 2^{ln} \left(1 + \frac{r}{\rho(y)} \frac{1}{2^l r}\right)^{-N} \|\delta_{2^l}(F^l)\|_{W_s^\infty}^2 \\ & \leq C 2^{ln} \min\{1, (2^l r)^N\} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty}^2, \end{aligned} \tag{4.17}$$

since $r \geq \rho(x_0)$, so there a constant C such that $\rho(y) \leq Cr$ for any $y \in B(x_0, r)$ by Lemma 2.1.

By (4.16) and (4.17), we get

$$\begin{aligned} & \|F^l(\sqrt{L})\|_{L^2(U_j(B)) \rightarrow L^\infty(B)} \\ & \leq \left(2^{-2sj} 2^{ln} (2^l r)^{-2s} \min\{1, (2^l r)^N\}\right)^{1/2} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty}. \end{aligned} \tag{4.18}$$

Now, we estimate $\|\bar{F}^l(\sqrt{L})\|_{L^2(U_j(B)) \rightarrow L^\infty(B)}$. The calculations symmetric to (4.16) with $\sup_{y \in B}$ replaced by $\sup_{z \in U_j(B)}$ yield. Then by Lemma 2.1, we have

$$\begin{aligned} & \|\bar{F}^l(\sqrt{L})\|_{L^2(U_j(B)) \rightarrow L^\infty(B)}^2 \\ & \leq C 2^{-2sj} 2^{ln} (2^l r)^{-2s} \sup_{z \in U_j(B)} \left(1 + \frac{r}{\rho(z)} \frac{1}{2^l r}\right)^{-N} \|\delta_{2^l}(F^l)\|_{W_s^\infty}^2 \\ & \leq C 2^{-2sj+jNl_0/(l_0+1)} 2^{ln} (2^l r)^{-2s} \min\{1, (2^l r)^N\} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty}^2. \end{aligned} \tag{4.19}$$

Combining (4.18) and (4.19), we get

$$\begin{aligned} & \left(\frac{1}{\Psi_\eta(B)|B|} \int_B |F(\sqrt{L})f_j|^{p_0} dy \right)^{1/p_0} \\ & \leq C 2^{-js+j(\eta+n)/p_0+jNl_0/2(l_0+1)} (2^l r)^{-s} \left(\min\{1, (2^l r)^N\} \max\{1, (2^l r)^{\frac{n}{2}}\} \right) \\ & \quad \times M_{\rho,\eta}(|f|^{p_0})^{1/p_0}(x) \sup_{l \in \mathbb{Z}} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty}. \end{aligned} \quad (4.20)$$

Therefore,

$$\begin{aligned} & \sum_{j=3}^{\infty} \sum_{l=-\infty}^{\infty} \left(\frac{1}{\Psi_\eta(B)|B|} \int_B |F^l(\sqrt{L})f_j|^{p_0} dy \right)^{1/p_0} \\ & \leq C \sum_{j=3}^{\infty} 2^{-js+j(\eta+n)/p_0+jNl_0/2(l_0+1)} \left(\sum_{l=-\infty}^{\infty} (2^l r)^{-s} \min\{1, (2^l r)^N\} \max\{1, (2^l r)^{\frac{n}{2}}\} \right) \\ & \quad \times M_{\rho,\eta}(|f|^{p_0})^{1/p_0}(x) \sup_{l \in \mathbb{Z}} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty} \\ & \leq C \sum_{j=3}^{\infty} 2^{-js+j(\eta+n)/p_0+jNl_0/2(l_0+1)} \left(\sum_{l:2^l r > 1} (2^l r)^{-s+\frac{n}{2}} + \sum_{l:2^l r \leq 1} (2^l r)^{N-s} \right) \\ & \quad \times M_{\rho,\eta}(|f|^{p_0})^{1/p_0}(x) \sup_{l \in \mathbb{Z}} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty} \\ & \leq C M_{\rho,\eta}(|f|^{p_0})^{1/p_0}(x) \sup_{l \in \mathbb{Z}} \|\delta_{2^l}[\varphi_l F]\|_{W_s^\infty}, \end{aligned} \quad (4.21)$$

if $N > s > n + \eta + N/2$.

Combining estimates (4.15) and (4.21), we have proved (4.9), and then estimate (3.5) holds. Note that $T = F(L)$ and $A_{r_B} = I - (I - e^{-r_B^2 L})^M$ commutes, it is easy to see that

$$\|A_{r_B} f\|_{L^\infty(c_0 B)} \leq C M_{\rho,\eta}(|f|^{p_0})^{\frac{1}{p_0}}(x) \quad \forall x \in c_0 B.$$

From this, (3.6) holds. Thus, (4.3) is proved.

Let us turn to prove (4.4). We will adapt some similar arguments in [17, 10]. We set $\omega \in A_1^{\rho,\eta}$ with $\eta > 0$. We know that $\omega \in A_1^{\rho,\eta} \subset A_p^{\rho,\eta}$ for every $1 < p < \infty$. Then $F(\sqrt{L})$ is bounded on $L^p(\omega)$ for $1 < p < \infty$. On the other hand, it is enough to prove the desired inequality for $f \in L_c^\infty$. If $\lambda > 0$, we consider the variant Calderón-Zygmund decomposition of f at level λ (ref [8]) and there exists a collection of balls $\{B_i\}$ such that $\{x \in \mathbb{R}^n : M_{\rho,\eta} f(x) > \tilde{c}\lambda\} = \bigcup_i B_i$, where \tilde{c} is a positive constant depending only on n .

Now we decompose f as $f = g + h = g + \sum_i h_i$, where

$$\begin{aligned} g(x) &= f(x) \chi_{\mathbb{R}^n \setminus \bigcup_i B_i} + \sum_i \left(\frac{1}{\Psi_\eta(B_i)|B_i|} \int_{B_i} f(y) \rho_i(y) dy \right) \chi_{B_i}(x), \\ h_i(x) &= f(x) \rho_i(x) - \left(\frac{1}{\Psi_\eta(B_i)|B_i|} \int_{B_i} f(y) \rho_i(y) dy \right) \chi_{B_i}(x), \end{aligned}$$

$$\rho_i(x) = \frac{\chi_{B_i}(x)}{\sum_j \chi_{B_j}(x)} \chi_{\cup_j B_j} \quad \text{and} \quad \chi_{\cup_j B_j} \leq C.$$

From these and the definition of $M_{\rho,\eta}f(x)$, we have

$$\lambda < \frac{1}{\Psi_\eta(B_i)|B_i|} \int_{B_i} |f(x)| dx \leq C_\eta \lambda,$$

If we denote $\Omega = \cup_i B_i := \cup_i B_i(x_i, r_i)$, then $|f(x)| \leq \lambda$ a.e. $x \in \mathbb{R}^n \setminus \Omega$, and $|g(x)| \leq C_\eta \lambda$ a.e. $x \in \mathbb{R}^n$. Observe that

$$\begin{aligned} \omega\{x \in \mathbb{R}^n : |F(\sqrt{L})f(x)| > \lambda\} &\leq \omega\{x \in \mathbb{R}^n : |F(\sqrt{L})g(x)| > \lambda/2\} \\ &\quad + \omega\{x \in \mathbb{R}^n : |F(\sqrt{L})h(x)| > \lambda/2\}. \end{aligned}$$

Since $F(\sqrt{L})$ is a bounded operator on $L^2(\omega)$. Then

$$\begin{aligned} \omega\{x \in \mathbb{R}^n : |F(\sqrt{L})g(x)| > \lambda/2\} &\leq \frac{C}{\lambda^2} \int_{\mathbb{R}^n} |g(x)|^2 \omega(x) dx \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |g(x)| \omega(x) dx \\ &\leq \frac{C}{\lambda} \left(\|f\|_{L^1(\omega)} + \int_{\cup_i B_i} |g(x)| \omega(x) dx \right) \\ &\leq \frac{C}{\lambda} \|f\|_{L^1(\omega)}. \end{aligned}$$

Let us deal with $F(\sqrt{L})h$. Set $t_i = r_i^2$ and write

$$h(x) = \sum_i h_i(x) = \sum_i (A_{t_i} h_i(x) + (h_i(x) - A_{t_i} h_i(x))),$$

where $A_{t_i} = e^{-t_i L}$. Note that for any $N > 2(l_0 + 1)\eta$, we have

$$\begin{aligned} |A_{t_i} h_i(x)| &\leq C_N t_i^{-n/2} \sup_{y \in B_i} e^{-b|x-y|^2/(t_i)} \int_{B_i} \left(1 + t_i/\rho^2(y)\right)^{-N} |b_i(y)| dy \\ &\leq C_N t_i^{-n/2} (1 + t_i/\rho^2(x_i))^{-N/(l_0+1)} \int_{B_i} \left(1 + t_i/\rho^2(y)\right)^{-N} |b_i(y)| dy \\ &\leq C_N t_i^{-n/2} (1 + t_i/\rho^2(x_i))^{-N/2(l_0+1)} \sup_{y \in B_i} e^{-b|x-y|^2/(4t_i)} \Psi_\eta(B_i)^{-1} \int_{B_i} |b_i(y)| dy \\ &\leq C_N \lambda |B_i| (1 + t_i/\rho^2(x_i))^{-N/2(l_0+1)} t_i^{-n/2} \inf_{y \in B_i} e^{-b|x-y|^2/(4t_i)} \\ &\leq C_N \lambda (1 + t_i/\rho^2(x_i))^{-N/2(l_0+1)} t_i^{-n/2} \int_{\mathbb{R}^n} e^{-b|x-z|^2/(4t_i)} \chi_{B_i}(z) dz. \end{aligned}$$

If $0 \leq u$ and $\|u\|_{L^2(\mathbb{R}^n)} = 1$, by the weak type (1,1) of $M_{\rho,\eta}$, we get

$$\int_{\mathbb{R}^n} \left(\sum_i |A_{t_i} b_i(x)| \right) u(x) dx \leq C \int_{\mathbb{R}^n} M_{\rho,\eta} u(y) \chi_{\cup_i B_i}(y) dy \leq \lambda^{1/2} \|f\|_{L^1(\mathbb{R}^n)}^{\frac{1}{2}}. \quad (4.22)$$

From this, we know that the above two series converge in $L^2(\mathbb{R}^n)$.

In addition, it is easy to see that

$$\begin{aligned} \omega\{x \in \mathbb{R}^n : |F(\sqrt{L})h(x)| > \lambda/2\} &\leq \omega\{x \in \mathbb{R}^n : |F(\sqrt{L})\left(\sum_i (A_{t_i}h_i)\right)(x)| > \lambda/4\} \\ &\quad + \omega\{x \in \mathbb{R}^n : |F(\sqrt{L})\left(\sum_i (h_i - A_{t_i}h_i)\right)(x)| > \lambda/4\}. \end{aligned}$$

Since T is bounded on $L^2(\omega)$, we have

$$\begin{aligned} \omega\{x \in \mathbb{R}^n : |F(\sqrt{L})\left(\sum_i (A_{t_i}h_i)\right)(x)| > \lambda/4\} &\leq \frac{4}{\lambda^2} \int_{\mathbb{R}^n} |F(\sqrt{L})\left(\sum_i (A_{t_i}h_i)\right)(x)|^2 \omega(x) dx \\ &\leq \frac{C}{\lambda^2} \left\| \sum_i |A_{t_i}h_i|^2 \omega^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

We consider $0 \leq u \in L_c^\infty(\mathbb{R}^n)$ with $\|u\|_{L^2} = 1$. We apply (4.22) to $u\omega^{\frac{1}{2}}$, by the weighted weak type (1,1) of $M_{\rho,\eta}$ (see Lemma 2.2), we then have

$$\begin{aligned} \int \left(\sum_i |A_{t_i}h_i|^2 \omega^{\frac{1}{2}} \right) u(x) dx &\leq C\lambda \int_{\mathbb{R}^n} M_{\rho,\eta}(u\omega^{\frac{1}{2}})(x) \chi_{\bigcup_i B_i}(x) dx \\ &\leq C\lambda \|M_{\rho,\eta}(u\omega^{\frac{1}{2}})\|_{L^2(\omega^{-1})} \|\chi_{\bigcup_i B_i}\|_{L^2(\omega)} \\ &\leq C\lambda^{\frac{1}{2}} \|f\|_{L^1(\omega)}. \end{aligned}$$

Hence,

$$\omega\left\{x \in \mathbb{R}^n : |F(\sqrt{L})\left(\sum_i A_{t_i}h_i\right)(x)| > \lambda/4\right\} \leq \frac{C}{\lambda} \|f\|_{L_\omega^1}. \quad (4.23)$$

On the other hand, set $\tilde{B}_i = 2B_i$, we then have

$$\begin{aligned} &\omega\left\{x \in \mathbb{R}^n : |F(\sqrt{L})\left(\sum_i (h_i - A_{t_i}h_i)\right)(x)| > \lambda/4\right\} \\ &\leq \omega\left(\bigcup_i \tilde{B}_i\right) + \omega\left\{x \in \mathbb{R}^n \setminus \bigcup_i \tilde{B}_i : |F(L)\left(\sum_i (h_i - A_{t_i}h_i)\right)(x)| > \lambda/4\right\} \\ &\leq \omega\left(\bigcup_i \tilde{B}_i\right) + \frac{4}{\lambda} \sum_i \int_{\mathbb{R}^n \setminus \tilde{B}_i} |F(\sqrt{L})(h_i - A_{t_i}h_i)(x)| \omega(x) dx \\ &:= I_1 + I_2. \end{aligned}$$

For I_1 , since $\omega \in A_1^{\rho,\eta}$, we get

$$\begin{aligned} I_1 &\leq C \sum_j \frac{\omega(\tilde{B}_j)}{\Psi_\eta(\tilde{B}_j)|\tilde{B}_j|} \varphi(B_j) |B_j| \\ &\leq \frac{C}{\lambda} \sum_j \frac{\omega(\tilde{B}_j)}{\Psi_\eta(\tilde{B}_j)|\tilde{B}_j|} \int_{B_j} |f(x)| dx \\ &\leq \frac{C}{\lambda} \sum_j \int_{B_j} |f(x)| M_{\rho,\eta} \omega(x) dx \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(x) dx. \end{aligned} \quad (4.24)$$

By Lemma 4.5, we then have

$$\begin{aligned}
& \int_{\mathbb{R}^n \setminus \tilde{B}_i} |F(\sqrt{L})(h_i - A_{t_i} h_i)(x)| \omega(x) dx \\
& \leq \int_{B_i} |h_i(y)| \int_{\mathbb{R}^n \setminus B(y, r_i)} |K_{F(1-A_{t_i})(\sqrt{L})}(x, y)| \omega(x) dx dy \\
& \leq C \int_{B_i} |h_i(y)| \omega(y) dy \\
& \leq C \int_{B_i} |f(y)| \omega(y) dy + C \lambda \omega(B_i).
\end{aligned} \tag{4.25}$$

Combining (4.24) and (4.25), we get

$$I_2 \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(x) dx.$$

Thus, (4.4) holds. \square

4.3. Proof of Theorem 4.2

We first prove (4.5). In fact, adapting the similar proof of (4.3), we can prove (3.7), (3.8) and (3.9) hold, if $\sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty} \leq C_s$ for s large enough. We omit the details here.

It remains to prove (4.6). We borrow some ideas from [19, 18]. We consider the case $k = 1$, the case $k \geq 2$ is similar. We give the same Calderón-Zygmund decomposition and g and h functions as in the proof of (4.4). Observe that $\sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty} \sim \sup_{t>0} \|\eta \delta_t F\|_{W_s^\infty}$, where $G(\lambda) = F(\sqrt{\lambda})$. We write $F_b(\sqrt{L})f(x) = F_b^1(\sqrt{L})f(x)$. Then,

$$\begin{aligned}
\omega\{x \in \mathbb{R}^n : |F_b(\sqrt{L})f(x)| > \lambda\} & \leq \omega\{x \in \mathbb{R}^n : |F_b(\sqrt{L})g(x)| > \lambda/2\} \\
& \quad + \omega\{x \in \mathbb{R}^n : |F_b(\sqrt{L})h(x)| > \lambda/2\}.
\end{aligned}$$

Since $F_b(\sqrt{L})$ is a bounded operator on $L^2(\omega)$. Then

$$\begin{aligned}
\omega\{x \in \mathbb{R}^n : |F_b(\sqrt{L})g(x)| > \lambda/2\} & \leq \frac{C}{\lambda^2} \int_{\mathbb{R}^n} |g(x)|^2 \omega(x) dx \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |g(x)| \omega(x) dx \\
& \leq \frac{C}{\lambda} \left(\|f\|_{L_\omega^1} + \int_{\bigcup_i B_i} |g(x)| \omega(x) dx \right) \leq \frac{C}{\lambda} \|f\|_{L_\omega^1}.
\end{aligned}$$

Let us deal with $F_b(\sqrt{L})h$. Set $t_i = r_i^2$ and write

$$h(x) = \sum_i h_i(x) = \sum_i (A_{t_i} h_i(x) + (h_i(x) - A_{t_i} h_i(x))).$$

Similar to the proof of (4.23), we have

$$\omega \left\{ x \in \mathbb{R}^n : |F_b(\sqrt{L}) \left(\sum_i A_{t_i} h_i \right) (x)| > \lambda/4 \right\} \leq \frac{C}{\lambda} \|f\|_{L^1(\omega)}.$$

In addition, set $\tilde{B}_i = 2B_i$ and $\tilde{\Omega} = \bigcup_i \tilde{B}_i$, we then have

$$\begin{aligned} & \omega \left\{ x \in \mathbb{R}^n : |F_b(\sqrt{L}) \left(\sum_i (h_i - A_{t_i} h_i) \right) (x)| > \lambda/4 \right\} \\ & \leq \omega(\tilde{\Omega}) + \omega \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |F_b(\sqrt{L}) \left(\sum_i (h_i - A_{t_i} h_i) \right) (x)| > \lambda/4 \right\} \\ & := II_1 + II_2. \end{aligned}$$

For II_1 , similar to the proof of (4.24), we get

$$II_1 \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(x) dx.$$

For II_2 , we have

$$\begin{aligned} II_2 & \leq \omega \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |F_b(\sqrt{L}) \left(\sum_{i \in E_1} (h_i - A_{t_i} h_i) \right) (x)| > \lambda/8 \right\} \\ & \quad + \omega \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |F_b(\sqrt{L}) \left(\sum_{i \in E_2} (h_i - A_{t_i} h_i) \right) (x)| > \lambda/8 \right\} \\ & \leq \omega \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : \left| \sum_{i \in E_1} (b - b_{B_i}) F(\sqrt{L}) ((h_i - A_{t_i} h_i)) (x) \right| > \lambda/16 \right\} \\ & \quad + \omega \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |F(\sqrt{L}) \left(\sum_{i \in E_1} (b - b_{B_i}) (h_i - A_{t_i} h_i) \right) (x)| > \lambda/16 \right\} \\ & \quad + \omega \left\{ x \in \mathbb{R}^n \setminus \tilde{\Omega} : |F_b(\sqrt{L}) \left(\sum_{i \in E_2} (h_i - A_{t_i} h_i) \right) (x)| > \lambda/8 \right\} \\ & := II_{21} + II_{22} + II_{23}, \end{aligned}$$

where $E_1 = \{i : r_i < \rho(x_i)\}$ and $E_2 = \{i : r_i \geq \rho(x_i)\}$, $B_i = B(x_i, r_i)$ and b_{B_i} is the average on B_i .

To estimate II_{21} , by Lemma 4.7, we have

$$\begin{aligned} II_{21} & \leq \frac{C}{\lambda} \sum_{i \in E_1} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |b(x) - b_{B_i}| |F(\sqrt{L}) ((h_i - A_{t_i} h_i)) (x)| \omega(x) dx \\ & \leq \frac{C}{\lambda} \sum_{i \in E_1} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |b(x) - b_{B_i}| \int_{B_i} |K_{F(I-A_{t_i})(\sqrt{L})}(x, y)| |h_i(y)| dy \omega(x) dx \\ & \leq \frac{C}{\lambda} \sum_{i \in E_1} \int_{B_i} |h_i(y)| \int_{\mathbb{R}^n \setminus \tilde{B}_i} |b(x) - b_{B_i}| |K_{F(I-A_{t_i})(\sqrt{L})}(x, y)| \omega(x) dx dy \\ & \leq \frac{C}{\lambda} \sum_{j \in E_1} \int_{B_i} |h_i(y)| \omega(y) dy \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(x) dx. \end{aligned}$$

For II_{22} , by (2.4) and (2.5), from Theorem 4.1 (b), we obtain

$$\begin{aligned}
II_{22} &\leq \frac{C}{\lambda} \sum_{i \in E_1} \int_{B_i} |b(x) - b_{B_i}| |h_j(x)| \omega(x) dx \\
&\leq \frac{C}{\lambda} \sum_{i \in E_1} \int_{B_i} |b(x) - b_{B_i}| |f(x)| \omega(x) dx \\
&\quad + \frac{C}{\lambda} \sum_{i \in E_1} \frac{1}{|B_i|} \int_{B_i} |f(y)| dy \int_{B_i} |b(x) - b_{B_i}| \omega(x) dx \\
&\leq \frac{C}{\lambda} \sum_{i \in E_1} \omega(B_i) \|b\|_{BMO_\theta(\rho)} \|f\|_{\log L, B_i, \omega} + \frac{C}{\lambda} \sum_{i \in E_1} \frac{\omega(B_i)}{|B_i|} \int_{B_i} |f(y)| dy \|b\|_{BMO_\theta(\rho)} \\
&\leq \frac{C}{\lambda} \sum_{i \in E_1} \omega(B_i) \|b - b_{B_i}\|_{\exp L, B_i, \omega} \left(\lambda + \frac{\lambda}{\omega(B_i)} \int_{B_i} \Phi \left(\frac{|f(y)|}{\lambda} \right) \omega(y) dy \right) \\
&\quad + \frac{C}{\lambda} \sum_{i \in E_1} \frac{\omega(B_i)}{|B_i|} \int_{B_i} |f(y)| dy \|b\|_{BMO_\theta(\rho)} \\
&\leq C \int_{\mathbb{R}^n} \Phi \left(\frac{|f(x)|}{\lambda} \right) \omega(x) dx.
\end{aligned}$$

For II_{23} . If $i \in E_2$, by Lemma 4.8, for any a ball $B_i = B(x_i, r_i)$, then the ball B can be decomposed into finite balls $\{Q_i^j\}_{j=1}^{j_i}$, there exists a constant L depending only on n such that at most L balls are disjoint each other, and $B \subset \bigcup_{j=1}^{j_i} Q_i^j \subset 2nB$ and $r_i/2 \leq \rho(x) \leq 2nC_0 r_i$ for some $x \in Q_i^j = Q(x_i^j, r_i^j)$. Let $h_i^j = h_i \chi_{Q_i^j \cap B_i}$, we have

$$\begin{aligned}
II_{23} &\leq \omega\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : \sum_{j \in E_2} \sum_{j=1}^{j_i} |b(x) - b_{Q_i^j}| |F(\sqrt{L})((h_i^j - A_{t_i} h_i^j)(x))| > \lambda/16\} \\
&\quad + \omega\{y \in \mathbb{R}^n \setminus \tilde{\Omega} : |F(\sqrt{L}) \left(\sum_{i \in E_2} \sum_{j=1}^{j_i} (b(x) - b_{Q_i^j}) h_i^j \right) (x)| > \lambda/16\} \\
&:= II_{23}^1 + II_{23}^2.
\end{aligned}$$

Similar to the proof of II_{23}^1 , note that $r_i^j \sim \rho(x_i^j)$, by Lemma 4.7, we then have

$$\begin{aligned}
II_{21} &\leq \frac{C}{\lambda} \sum_{i \in E_2} \sum_{j=1}^{j_i} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |b(x) - b_{Q_i^j}| |F(\sqrt{L})((h_i^j - A_{t_i} h_i^j)(x))| \omega(x) dx \\
&\leq \frac{C}{\lambda} \sum_{i \in E_2} \sum_{j=1}^{j_i} \int_{\mathbb{R}^n \setminus \tilde{\Omega}} |b(x) - b_{Q_i^j}| \int_{Q_i^j} |K_{F(I-A_{t_i})(\sqrt{L})}(x, y)| |h_i^j(y)| dy \omega(x) dx \\
&\leq \frac{C}{\lambda} \sum_{i \in E_2} \sum_{j=1}^{j_i} \int_{Q_i^j} |h_i^j(y)| \int_{\mathbb{R}^n \setminus 2Q_i^j} |b(x) - b_{Q_i^j}| |K_{F(I-A_{t_i})(\sqrt{L})}(x, y)| \omega(x) dx dy \\
&\leq \frac{C}{\lambda} \sum_{i \in E_2} \sum_{j=1}^{j_i} \int_{Q_i^j \cap B_i} |h_i(y)| \omega(y) dy \leq \frac{C}{\lambda} \sum_{i \in E_2} \int_{B_i} |h_i(y)| \omega(y) dy
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{\lambda} \sum_{i \in E_2} \int_{B_i} |f(y)| \omega(y) dy + \frac{C}{\lambda} \sum_{i \in E_2} \frac{\omega(B_i)}{\Psi_\eta(B_i) |B_i|} \int_{B_i} |f(y)| dy \\
&\leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(x) dx.
\end{aligned}$$

Similar to the proof of II_{22} , set $f_i^j = f \chi_{Q_i^j \cap B_i}$, note that $r_i^j \sim \rho(x_i^j)$, we then have

$$\begin{aligned}
II_{23}^2 &\leq \frac{C}{\lambda} \sum_{i \in E_2} \sum_{j=1}^{j_i} \int_{Q_i^j} |b(x) - b_{Q_i^j}| |h_i^j(x)| \omega(x) dx \\
&\leq \frac{C}{\lambda} \sum_{i \in E_2} \sum_{j=1}^{j_i} \int_{Q_i^j} |b(x) - b_{Q_i^j}| |f_i^j(x)| \omega(x) dx \\
&\quad + \frac{C}{\lambda} \sum_{i \in E_2} \sum_{j=1}^{j_i} \frac{1}{\Psi_\eta(B_i) |B_i|} \int_{B_i} |f(y)| dy \int_{Q_i^j} |b(x) - b_{Q_i^j}| \omega(x) dx \\
&\leq \frac{C}{\lambda} \sum_{i \in E_2} \sum_{j=1}^{j_i} \|b\|_{BMO_\theta(\rho)} \omega(Q_i^j) \|f_i^j\|_{\log L, Q_i^j, \omega} \\
&\quad + \frac{C}{\lambda} \sum_{i \in E_2} \sum_{j=1}^{j_i} \frac{\omega(Q_i^j)}{\Psi_\eta(B_i) |B_i|} \int_{B_i} |f(y)| dy \|b\|_{BMO_\theta(\rho)} \\
&\leq \frac{C}{\lambda} \sum_{i \in E_2} \sum_{j=1}^{j_i} \omega(Q_i^j) \left(\lambda + \frac{\lambda}{\omega(Q_i^j)} \int_{Q_i^j \cap B_i} \Phi(|f(y)|/\lambda) \omega(y) dy \right) \\
&\quad + \frac{C}{\lambda} \sum_{i \in E_2} \frac{L\omega(2nB_i)}{\Psi_\eta(B_i) |B_i|} \int_{B_i} |f(y)| dy \\
&\leq C \sum_{i \in E_2} \left(\omega(2nB_i) + \int_{B_i} \Phi(|f(y)|/\lambda) \omega(y) dy + \frac{1}{\lambda} \int_{B_i} |f(y)| \omega(y) dy \right) \\
&\leq C \int_{\mathbb{R}^n} \Phi \left(\frac{|f(x)|}{\lambda} \right) \omega(x) dx.
\end{aligned}$$

In the last inequality we used the following fact that (see the proof of (4.24))

$$\sum_{i \in E_2} \omega(2nB_i) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(x) dx.$$

Thus, (4.6) holds. \square

5. Littlewood-Paley operators

Let L be the same as Section 4. We define the Littlewood-Paley operator for $x \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$,

$$g_L f(x) = \left(\int_0^\infty (tL)^{1/2} e^{-tL} f(x)^2 \frac{dt}{t} \right)^{1/2}.$$

We have the following L^p estimates.

Lemma 5.1. *Let $1 < p < \infty$, then*

$$\|g_L f\|_{L^p} \sim \|f\|_{L^p}, \quad \forall f \in L^p \cap L^2.$$

Lemma 5.1 is a special case in [1].

We have the following weighted estimates for Littlewood-Paley operators.

Theorem 5.1. *Let g_L be defined as above.*

(a) *If $1 < p < \infty$ and $\omega \in A_p^{\rho, \infty}$, then*

$$\|g_L f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}, \quad \forall f \in L_c^\infty,$$

(b) *If $1 < p < \infty$ and $\omega \in A_p^{\rho, \infty}$, then*

$$\|f\|_{L^p(\omega)} \leq C \|g_L f\|_{L^p(\omega)}, \quad \forall f \in L^p \cap L^2,$$

(c) *If $\omega \in A_1^{\rho, \infty}$, then*

$$\|g_L f\|_{L^{1, \infty}(\omega)} \leq C \|f\|_{L^1(\omega)}, \quad \forall f \in L_c^\infty.$$

We also define the commutator for the Littlewood-Paley operator for $x \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$,

$$g_{L,b}^k f(x) = \left(\int_0^\infty (tL)^{1/2} e^{-tL} (b(x) - b(\cdot))^k f(x) \right)^2 \frac{dt}{t} \Big)^{1/2}.$$

We give the following weighted estimates the commutator for the Littlewood-Paley operator.

Theorem 5.2. *Let $k \in \mathbb{N}$ and $b \in BMO_\theta(\rho)$.*

(a) *If $1 < p < \infty$ and $\omega \in A_p^{\rho, \infty}$, then*

$$\|g_{L,b}^k f\|_{L^p(\omega)} \leq C \|b\|_{BMO_\theta(\rho)}^k \|f\|_{L^p(\omega)},$$

(b) *If $\omega \in A_1^{\rho, \infty}$, then*

$$\omega(\{x \in \mathbb{R}^n : |g_{L,b}^k f(x)| > \lambda\}) \leq C \Phi(\|b\|_{BMO_\theta(\rho)}) \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) \omega(x) dx,$$

where $\Phi(t) = t \log(e + t)^k$.

Before we begin the arguments, we recall some basic facts about Hilbert-valued extensions of scalar inequalities. To do so we introduce some notation: by \mathcal{H} we mean $L^2((0, \infty), \frac{dt}{t})$ and $\|\cdot\|$ denotes the norm in \mathcal{H} . Hence, for a function $h : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C}$, we have for $x \in \mathbb{R}^n$

$$\|\mathcal{H}(x, \cdot)\| = \left(\int_0^\infty |h(x, t)|^2 \frac{dt}{t} \right)^{1/2}.$$

In particular,

$$g_L f(x) = \|\varphi(L, \cdot) f(x)\|$$

and

$$g_{L,b}^k f(x) = \|\varphi(L, \cdot) (b(x) - b(\cdot))^k f(x)\|$$

with $\varphi(z, t) = (tz)^{1/2} e^{-tz}$. Let $L_{\mathcal{H}}^p(\omega)$ be the space of \mathcal{H} -valued $L^p(\omega)$ -functions equipped with the norm

$$\|h\|_{L_{\mathcal{H}}^p(\omega)} = \left(\int_{\mathbb{R}^n} \|h(x, \cdot)\|^p d\omega(x) \right)^{1/p}.$$

Lemma 5.2([3]). *Let μ be a Borel measure on \mathbb{R}^n . Let $1 \leq p \leq q < \infty$. Let D be a subspaces of \mathcal{M} , the space of measurable functions in \mathbb{R}^n . Let S, T be linear operators from \mathcal{D} into \mathcal{M} . Assume there exists $C_1 > 0$ such that for all $f \in \mathcal{D}$, we have*

$$\|Tf\|_{L^q(\mu)} \leq C_1 \sum_{j \geq 1} \alpha_j \|Sf\|_{L^q(F_j, \mu)},$$

where F_j are subsets of \mathbb{R}^n and $\alpha_j \geq 0$. Then, there is an \mathcal{H} -valued extension with the same constant: for all $f : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C}$ such that for (almost) all $t > 0, f(\cdot, t) \in \mathcal{D}$,

$$\|Tf\|_{L^q_{\mathcal{H}}(\mu)} \leq C_1 \sum_{j \geq 1} \alpha_j \|Sf\|_{L^q_{\mathcal{H}}(F_j, \mu)}.$$

5.2. Proof of Theorem 5.1(a)

Since $\omega \in A_p^{\rho, \infty}$, there exist $1 < p_0 < p$ and θ such that $\omega \in A_{p/p_0}^{\rho, \theta}$. We are going to apply Theorem 3.2 with $k = 0, T = g_L, A_r = I - (I - e^{-r^2 L})^M, M \in \mathbb{N}$ large enough. We first show (3.9) holds for all $f \in L_c^\infty$ and any $\eta > 0$.

Let $1 \leq m \leq M$. If $B = B(x_0, r)$ with $r < \rho(x_0), \bar{B} = c_0 B, B_j = 2^{j+1} \bar{B}, C_j(\bar{B}) = B_j \setminus B_{j-1}$ for $j \geq 1$, and $g \in L^{p_0}$ with $\text{supp } g \subset C_j(\bar{B})$, by (4.1), we have

$$\|e^{-mr^2 L} g\|_{L^\infty(\bar{B})} \leq C_1 2^{j(n+\eta)} e^{-\alpha 4^j} \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{C_j(\bar{B})} |g|^{p_0} dx \right)^{1/p_0}. \quad (5.1)$$

Lemma 5.2 applied to $S = I, T : L^{p_0} = L^{p_0}(\mathbb{R}^n) \rightarrow L^{q_0}(\mathbb{R}^n)$ given by

$$Tg = (C_1 2^{j(n+\theta)} e^{-\alpha 4^j})^{-1} \frac{|B_j|^{1/p_0}}{|\bar{B}|^{1/q_0}} \chi_{\bar{B}} e^{mr^2 L} (\chi_{C_j(\bar{B})} g)$$

yields for any q_0 satisfying $p_0 < q_0 < \infty$

$$\left(\frac{1}{|B_j|} \int_{\bar{B}} \|e^{-mr^2 L} g(x, \cdot)\|^{q_0} dx \right)^{1/q_0} \leq C_1 2^{j(n+\eta)} e^{-\alpha 4^j} \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{C_j(\bar{B})} |g|^{p_0} dx \right)^{1/p_0}. \quad (5.2)$$

Since (5.2) holds for any $p_0 < q_0 < \infty$, so

$$\|e^{-mr^2 L} g(x, \cdot)\|_{L^\infty(\bar{B})} \leq C_1 2^{j(n+\eta)} e^{-\alpha 4^j} \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{C_j(\bar{B})} |g|^{p_0} dx \right)^{1/p_0}. \quad (5.3)$$

for all $g \in L_{\mathcal{H}}^{p_0}$ with $\text{supp } g(x, \cdot) \subset C_j(\bar{B})$ for each $t > 0$ and some $\alpha > 0$.

For $h \in L_{\mathcal{H}}^{p_0}$, we write $h(x, t) = \sum_{j \geq 1} h_j(x, t), x \in \mathbb{R}^n, t > 0$, where $h_j(x, t) = h(x, t) \chi_{C_j(B)}(x)$. Using (5.3), we have for $1 \leq m \leq M$,

$$\begin{aligned} \|\|e^{-mr^2 L} h(x, \cdot)\|\|_{L^\infty(\bar{B})} &\leq \sum_{j \geq 1} \|\|e^{-mr^2 L} g(x, \cdot)\|\|_{L^\infty(\bar{B})} \\ &\leq C \sum_{j \geq 1} 2^{j(n+\eta)} e^{-\alpha 4^j} \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{C_j(\bar{B})} |h|^{p_0} dx \right)^{1/p_0}. \end{aligned} \quad (5.4)$$

Take $h(x, t) = (tL)^{1/2}e^{-tL}f(x)$. Since $g_L f(x) = \|h(x, \cdot)\|$ and $f \in L_c^\infty$, $h \in L_{\mathcal{H}}^{p_0}$ by Lemma 5.1 and

$$g_L(e^{-mr^2L}f)(x) = \left(\int_0^\infty |(tL)^{1/2}e^{-tL}e^{-mr^2L}f(x)|^2 \frac{dt}{t} \right)^{1/2} = \|e^{-mr^2L}h(x, \cdot)\|.$$

Thus, (5.4) implies

$$\|g_L(e^{-mr^2L}f)\|_{L^\infty(\bar{B})} \leq C \sum_{j \geq 1} 2^{j(n+\eta)} e^{-\alpha 4^j} \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{B_j} |g_L f|^{p_0} dx \right)^{1/p_0}.$$

and it follows that g_L satisfies (3.9).

We now show (3.7) with $k = 0$ holds for all $f \in L_c^\infty$ and any $\eta > 0$. Set $B = B(x_0, r)$ with $r < \rho(x_0)$, $\bar{B} = c_0 B$. Write $f = \sum_{j \geq 1} f_j$ as before. If $j = 1$ we use that both g_L and $(I - e^{-r^2L})^M$ are bounded on L^{p_0} , then

$$\left(\frac{1}{|\bar{B}|} \int_{\bar{B}} |g_L(I - e^{-r^2L})^M f_1|^{p_0} dx \right)^{1/p_0} \leq C \left(\frac{1}{|4\bar{B}|} \int_{4\bar{B}} |f|^{p_0} dx \right)^{1/p_0}. \quad (5.5)$$

For $j \geq 2$, we observe that

$$\begin{aligned} g_L((I - e^{-r^2L})^M f_j)(x) &= \left(\int_0^\infty |(tL)^{1/2}e^{-tL}(I - e^{-r^2L})^M f_j(x)|^2 \frac{dt}{t} \right)^{1/2} \\ &= \|\varphi(L, \cdot) f_j(x, \cdot)\|, \end{aligned}$$

where $\varphi(z, t) = (tz)^{1/2}e^{-tz}(1 - e^{-r^2z})^M$. Then $\varphi(z, t)$ is a holomorphic function in $\sum_\mu = \{z \in \mathbb{C}^* : |\arg z| < \mu\}$ with $\mu \in (\nu, \pi)$, where $\nu \in [0, \pi/2)$ is defined by

$$\nu = \sup\{|\arg z| : z \in \mathcal{D}(L)\}.$$

Assume that $\nu < \theta < \nu < \mu < \pi/2$. As in [1], we then have

$$\varphi(L, t) = \int_{\Gamma_+} e^{-zL} \eta_+(z, t) dz + \int_{\Gamma_-} e^{-zL} \eta_-(z, t) dz,$$

where Γ_\pm is the half-ray $\mathbb{R}^+ e^{\pm i(\pi/2 - \theta)}$,

$$\eta_\pm(z, t) = \frac{1}{2\pi i} \int_\pm e^{\xi z} \varphi(\xi, t) d\xi, \quad z \in \Gamma_\pm,$$

where γ_\pm being the half-ray $\mathbb{R}^+ e^{\pm i\nu}$ and Γ_\pm is the half-ray $\mathbb{R}^+ e^{\pm i(\pi/2 - \theta)}$. Note that

$$|\eta_\pm(z, t)| \leq C \frac{t^{1/2}}{(|z| + t)^{3/2}} \frac{r^{2M}}{(|z| + t)^M}, \quad z \in \Gamma_\pm, t > 0.$$

Thus,

$$\|\eta_\pm(z, \cdot)\| \leq \left(\int_0^\infty \frac{t^{1/2}}{(|z| + t)^{3/2}} \frac{r^{2M}}{(|z| + t)^M} \frac{dt}{t} \right)^{1/21} \leq \frac{r^{2M}}{|z|^{M+1}}.$$

Applying Minkowski's inequality, by (4.1), we get

$$\begin{aligned}
& \left(\frac{1}{|\bar{B}|} \int_{\bar{B}} \left\| \int_{\Gamma_+} e^{-zL} f_j(x) \eta_+(z, \cdot) dz \right\|^{p_0} dx \right)^{1/p_0} \\
& \leq \left(\frac{1}{|\bar{B}|} \int_{\bar{B}} \left(\int_{\Gamma_+} |e^{-zL} f_j(x)| \|\eta_+(z, \cdot)\| |dz| \right)^{p_0} dx \right)^{1/p_0} \\
& \leq \int_{\Gamma_+} \left(\frac{1}{|\bar{B}|} \int_{\bar{B}} |e^{-zL} f_j(x)|^{p_0} dx \right)^{1/p_0} \frac{r^{2M}}{|z|^{M+1}} |dz| \\
& \leq C 2^{j(n+\eta)} \int_0^\infty \left(\frac{r}{\sqrt{s}} \right)^{n/2} e^{-\alpha \frac{4j r^2}{s}} \frac{r^M}{s^{M+1}} ds \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{C_j(\bar{B})} |f|^{p_0} dy \right)^{1/p_0} \\
& \leq C 2^{j(n+\eta-2M)} \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{B_j} |f|^{p_0} dy \right)^{1/p_0}
\end{aligned}$$

provided $2M > n + \eta$. This plus the corresponding term for Γ_- yield

$$\left(\frac{1}{|\bar{B}|} \int_{\bar{B}} |g_L(I - e^{-r^2 L})^M f_j|^{p_0} dx \right)^{1/p_0} \leq C 2^{j(n+\eta-2M)} \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{B_j} |f|^{p_0} dy \right)^{1/p_0}. \quad (5.6)$$

Combining (5.5) and (5.6), we obtain (3.7) holds whenever $2M > n + \eta$.

We now show (3.8) with $k = 0$ holds for all $f \in L_c^\infty$ and any $\eta > 0$. Set $B = B(x_0, r)$ with $r \geq \rho(x_0)$. Write $f = \sum_{j \geq 1} f_j = \sum_{j \geq 1} f \chi_{C_j(B)}$, $B_j = 2^{j+1}B$ and $C_j(B) = B_j \setminus B_{j-1}$.

$$\left(\frac{1}{\Psi_\eta(B)|B|} \int_B |g_L f_1|^{p_0} dx \right)^{1/p_0} \leq C \left(\frac{1}{\Psi_\eta(4B)|4B|} \int_{4B} |f|^{p_0} dx \right)^{1/p_0}. \quad (5.7)$$

For $j \geq 2$, we observe that

$$g_L(f_j)(x) = \left(\int_0^\infty |(tL)^{1/2} e^{-tL} f_j(x)|^2 \frac{dt}{t} \right)^{1/2} = \|\varphi(L, \cdot) f_j(x, \cdot)\|,$$

where $\varphi(z, t) = (tz)^{1/2} e^{tz}$. The functions $\eta_\pm(\cdot, t)$ associated with $\varphi(\cdot, t)$ verify

$$|\eta_\pm(z, t)| \leq C \frac{t^{1/2}}{(|z| + t)^{3/2}}, \quad z \in \Gamma_\pm, t > 0.$$

From this, note that $r \geq \rho(x_0)$, By (4.1) and Lemma 2.1, we then have

$$\begin{aligned}
& \left(\frac{1}{\Psi_\eta(B)|B|} \int_B \left\| \int_{\Gamma_+} e^{-zL} f_j(x) \eta_+(z, \cdot) dz \right\|^{p_0} dx \right)^{1/p_0} \\
& \leq \left(\frac{1}{\Psi_\eta(B)|B|} \int_B \left(\int_{\Gamma_+} |e^{-zL} f_j(x)| \|\eta_+(z, \cdot)\| |dz| \right)^{p_0} dx \right)^{1/p_0} \\
& \leq \int_{\Gamma_+} \left(\frac{1}{\Psi_\eta(B)|B|} \int_B |e^{-zL} f_j(x)|^{p_0} dx \right)^{1/p_0} \frac{1}{|z|} |dz| \\
& \leq C 2^{j(n+\eta)} \int_0^\infty \left(\frac{1}{\sqrt{s}} \right)^n e^{-\alpha \frac{4j r^2}{s}} \int_B \left(1 + \frac{\sqrt{s}}{\rho(x)} \right)^{-N} dx \frac{1}{s} ds \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{C_j(B)} |f|^{p_0} dy \right)^{1/p_0}
\end{aligned}$$

$$\begin{aligned}
&\leq C2^{j(n+\eta)} \int_0^\infty \left(\frac{1}{\sqrt{s}}\right)^n e^{-\alpha \frac{4^j r^2}{s}} \int_B \left(1 + \left(\frac{r}{\rho(x_0)}\right)^{l_0/(l_0+1)} \frac{\sqrt{s}}{r}\right)^{-N} dx \frac{1}{s} ds \\
&\quad \times \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{C_j(B)} |f|^{p_0} dy\right)^{1/p_0} \\
&\leq C2^{j(n+\eta)} \int_0^\infty \left(\frac{r}{\sqrt{s}}\right)^n e^{-\alpha \frac{4^j r^2}{s}} \min\left\{\left(\frac{r}{\sqrt{s}}\right)^N, 1\right\} \frac{1}{s} ds \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{C_j(B)} |f|^{p_0} dy\right)^{1/p_0} \\
&\leq C2^{j(n+\eta)} \left(\int_0^{r^2} \left(\frac{r}{\sqrt{s}}\right)^n \left(\frac{s}{4^j r^2}\right)^N \frac{1}{s} ds + \int_{r^2}^\infty \left(\frac{r}{\sqrt{s}}\right)^{n+N} \left(\frac{s}{4^j r^2}\right)^{N/2} \frac{1}{s} ds\right) \\
&\quad \times \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{C_j(B)} |f|^{p_0} dy\right)^{1/p_0} \\
&\leq C2^{j(n+\eta-N)} \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{B_j} |f|^{p_0} dy\right)^{1/p_0}
\end{aligned}$$

provided $N > n + \eta$. This plus the corresponding term for Γ_- yield

$$\left(\frac{1}{\Psi_\eta(B)|B|} \int_B |g_L f|^{p_0} dx\right)^{1/p_0} \leq C2^{j(n+\eta-N)} \left(\frac{1}{\Psi_\eta(B_j)|B_j|} \int_{B_j} |f|^{p_0} dy\right)^{1/p_0}. \quad (5.8)$$

Combining (5.7) and (5.8), we obtain (3.8) holds whenever $N > n + \eta$. \square

5.3. Proof of Theorem 5.1(b)

To prove Theorem 5.1 (b), we introduce the following operator. Define for $f \in L^2_{\mathcal{H}}$ and $x \in \mathbb{R}^n$,

$$T_L f(x) = \int_0^\infty (tL)^{1/2} e^{-tL} f(x, t) \frac{dt}{t}.$$

Recall that $(tL)^{1/2} e^{-tL} f(x, t) = (tL)^{1/2} e^{-tL} (f(\cdot, t))(x)$. Hence, T_L maps \mathcal{H} -valued functions to \mathbb{C} -valued functions. For $f \in L^2_{\mathcal{H}}$ and $h \in L^2$, we have

$$\int_{\mathbb{R}^n} T_L f \bar{h} dx = \int_{\mathbb{R}^n} \int_0^\infty f(x, t) \overline{(tL^*)^{1/2} e^{-tL^*} h(x)} \frac{dt}{t} dx,$$

where L^* is the adjoint of L , hence,

$$\left| \int_{\mathbb{R}^n} T_L f \bar{h} dx \right| \leq \int_{\mathbb{R}^n} \|f(x, \cdot)\| g_{L^*}(h)(x) dx.$$

Since g_{L^*} is bounded on L^p for $1 < p < \infty$. This and a density imply that T_L has a bounded extension from $L^p_{\mathcal{H}}$ to L^p . We next give a weighted result for the operator T_L .

Lemma 5.3. *Let $1 < p < \infty$ and $\omega \in A_p^{\rho, \infty}$, then for all $f \in L_c^\infty(\mathbb{R}^n \times (0, \infty))$ we have*

$$\|T_L f\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)}.$$

Hence, T_L has a bounded extension from $L^p_{\mathcal{H}}(\omega)$ to $L^p(\omega)$.

Proof. We will apply Theorem 3.2 with $k = 0$ (its vector-valued extension) with underlying measure dx and weight ω to linear operator $T = T_L$ and $A_r = I - (I - e^{-r^2 L})^M$, $M \in \mathbb{N}$ large enough. Adapting similar to the proof of Theorem 5.1, we can obtain the desired result. We omit the details here. \square

Proof of Theorem 5.1 (b). Let $f \in L^2$ and define F by $F(x, t) = (tL)^{1/2} e^{-tL} f(x)$. Note that $F \in L^2_{\mathcal{H}}$ since $\|F\|_{L^2_{\mathcal{H}}} = \|g_L f\|_{L^2}$. By functional calculus on L^2 , we have

$$f = 2 \int_0^\infty (tL)^{1/2} e^{-tL} F(\cdot, t) \frac{dt}{t} = 2T_L F \quad (5.11)$$

with convergence in L^2 . Note that e^{-tL} has an infinitesimal generator on $L^p(\omega)$ for $1 < p < \infty$. Let us call $L_{p,\omega}$ this generator. In particular e^{-tL} and $e^{-tL_{p,\omega}}$ agree on $L^p(\omega) \cap L^2$. Our assert that $L_{p,\omega}$ has a bounded holomorphic functional calculus on $L^p(\omega)$, hence replacing L by $L_{p,\omega}$ and $f \in L^2$ by $f \in L^p(\omega)$, we see that $F \in L^p_{\mathcal{H}}(\omega)$ with $\|F\|_{L^p(\omega)} = \|g_{L_{p,\omega}} f\|_{L^p(\omega)}$ and (5.11) is valid with convergence in $L^p(\omega)$. Thus, by Lemma 5.3,

$$\|f\|_{L^p(\omega)} = 2\|T_{L_{p,\omega}} F\|_{L^p(\omega)} \leq C\|F\|_{L^p_{\mathcal{H}}(\omega)} = \|g_{L_{p,\omega}} f\|_{L^p(\omega)}.$$

Note that $g_L f = g_{L_{p,\omega}} f$ when $f \in L^2 \cap L^p(\omega)$ and $T_L = T_{L_{p,\omega}}$ when $F \in L^2_{\mathcal{H}} \cap L^p_{\mathcal{H}}(\omega)$. \square

5.4. Proof of Theorem 5.1(c)

To prove Theorem 5.1(c), it suffices to show the following Lemma.

Lemma 5.4. *Suppose $\omega \in A_1^{\rho, \theta}$ and $B = B(x_0, r)$. Then for $\text{supp } f \subset B$,*

$$\int_{\mathbb{R}^n \setminus 2B(x_0, r)} |g_L(1 - e^{-r^2 L}) f(x)| \omega(x) dx \leq C \int_B |f(y)| \omega(y) dy. \quad (5.9)$$

Proof. In fact,

$$\begin{aligned} \int_{\mathbb{R}^n \setminus 2B(x_0, r)} |g_L(1 - e^{-r^2 L}) f(x)| \omega(x) dx &\leq \int_{\mathbb{R}^n \setminus 2B(x_0, r)} \left\| \int_{\Gamma_+} e^{-zL} f(x) \eta_+(z, \cdot) dz \right\| \omega(x) dx \\ &\quad + \int_{\mathbb{R}^n \setminus 2B(x_0, r)} \left\| \int_{\Gamma_-} e^{-zL} f(x) \eta_+(z, \cdot) dz \right\| \omega(x) dx \\ &:= I_1 + I_2. \end{aligned}$$

We only give the estimate for I_1 , I_2 is similar. Then for any N large enough, we have

$$\begin{aligned} I_1 &\leq \int_{\mathbb{R}^n \setminus 2B(x_0, r)} \int_{\Gamma_+} |e^{-zL} f(x)| \|\eta_+(z, \cdot)\| |dz| \omega(x) dx \\ &\leq \int_{\Gamma_+} \int_{\mathbb{R}^n \setminus 2B(x_0, r)} |e^{-zL} f(x)| \omega(x) dx \frac{r^2}{|z|^2} |dz| \\ &\leq C \int_0^\infty \int_{\mathbb{R}^n \setminus 2B(x_0, r)} \int_B \left(1 + \frac{\sqrt{s}}{\rho(y)}\right)^{-N} s^{-n/2} e^{-\alpha|x-y|^2/2s} |f(y)| dy \omega(x) dx \frac{r^2}{s^2} ds \\ &\leq C \int_B \int_0^\infty \int_{\mathbb{R}^n \setminus B(y, r)} s^{-n/2} \left(1 + \frac{\sqrt{s}}{\rho(y)}\right)^{-N} e^{-\alpha|x-y|^2/2s} \omega(x) dx e^{-\alpha r^2/2s} \frac{r^2}{s^2} ds |f(y)| dy \\ &\leq C \int_B \int_0^\infty e^{-\alpha r^2/2s} \frac{r^2}{s^2} ds |f(y)| \omega(y) dy \leq C \int_B |f(y)| \omega(y) dy. \end{aligned}$$

Thus, (5.9) holds. \square

5.5. Proof of Theorem 5.2

We first prove Theorem 5.2(a). In fact, adapting the similar proof of Theorem 5.1(a), we can prove (3.7), (3.8) and (3.9) hold. We omit the details here.

To prove Theorem 5.2(b), it suffices to show the following Lemma.

Lemma 5.5. *Suppose $b \in BMO_\theta(\rho)$, $\omega \in A_1^{\rho, \theta_1}$, and $B = B(x_0, r)$ with $r < \rho(x_0)$. Then for any $f \in L^1(\omega)$ and $\text{supp } f \subset B$*

$$\int_{\mathbb{R}^n \setminus 2B(x_0, r)} |(b(x) - b_B)g_L(1 - e^{-r^2 L})f(x)|\omega(x)dx \leq C \int_B |f(y)|\omega(y)dy. \quad (5.10)$$

Proof. In fact,

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus 2B(x_0, r)} |(b(x) - b_B)g_L(1 - e^{-r^2 L})f(x)|\omega(x)dx \\ & \leq \int_{\mathbb{R}^n \setminus 2B(x_0, r)} \left\| \int_{\Gamma_+} (b(x) - b_B)e^{-zL}f(x)\eta_+(z, \cdot)dz \right\| \omega(x)dx \\ & \quad + \int_{\mathbb{R}^n \setminus 2B(x_0, r)} \left\| \int_{\Gamma_-} (b(x) - b_B)e^{-zL}f(x)\eta_+(z, \cdot)dz \right\| \omega(x)dx \\ & := I_1 + I_2. \end{aligned}$$

We only give the estimate for I_1 , I_2 is similar. Then for any N large enough, we have

$$\begin{aligned} I_1 & \leq \int_{\mathbb{R}^n \setminus 2B(x_0, r)} \int_{\Gamma_+} |e^{-zL}f(x)|\|\eta_+(z, \cdot)\| |dz| \omega(x)dx \\ & \leq \int_{\Gamma_+} \int_{\mathbb{R}^n \setminus 2B(x_0, r)} |e^{-zL}f(x)|\omega(x)dx \frac{r^2}{|z|^2} |dz| \\ & \leq C \int_0^\infty \int_{\mathbb{R}^n \setminus 2B(x_0, r)} \int_B \left(1 + \frac{\sqrt{s}}{\rho(y)}\right)^{-N} s^{-n/2} (b(x) - b_B) e^{-\alpha|x-y|^2/2s} \\ & \quad \times |f(y)| dy \omega(x) dx \frac{r^2}{s^2} ds \\ & \leq C \int_B \int_0^\infty \int_{\mathbb{R}^n \setminus B(y, r)} s^{-n/2} \left(1 + \frac{\sqrt{s}}{\rho(y)}\right)^{-N} (b(x) - b_B) e^{-\alpha|x-y|^2/2s} \omega(x) dx \\ & \quad \times e^{-\alpha r^2/2s} \frac{r^2}{s^2} ds |f(y)| dy \\ & \leq C \|b\|_{BMO_\theta(\rho)} \int_B \int_0^\infty e^{-\alpha r^2/2s} \frac{r^2}{s^2} ds |f(y)| \omega(y) dy \\ & \leq C \|b\|_{BMO_\theta(\rho)} \int_B |f(y)| \omega(y) dy. \end{aligned}$$

Thus, (5.10) holds. \square

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